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Hahn Banach Theorem On Fuzzy Soft Normed Linear Space

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ARTICLE INFO | ABSTRACT

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Abstract

In this paper the fundamental definitions like linear operator, norm of a linear operator, continuityand boundedness have been extended into fuzzy soft settings in a different notion. Also some theorems related to these concepts have been proved and the famous Hahn Banach theorem is established.

KEYWORDS:. Fuzzy soft map, fuzzy soft linear operator, norm of a fuzzy soft operator ,fuzzy soft bounded operator, fuzzy soft continuous operator, fuzzy soft linear functional

1. Introduction

Zadeh [6] in 1965 introduced the concept of fuzzy set, it plays a vital role in studying problems in real life. In 1999, Molodostov [2] introduced the concept of soft set which is another mathematical tool dealing with uncertainties. Maji et al. [1] in 2001 combined the fuzzy and soft sets that developed the new theory of fuzzy soft sets. Sujoy Das, Samanta have introduced the idea of soft linear functionals over soft linear space.

In this paper the concept of linear operator on fuzzy soft linear space is coined, with the help of it the famous Hahn Banach theorem is furnished in fuzzy soft setting. Further boundedness and continuity of fuzzy soft linear operator are defined. Later important result are developed relating to these concepts.

2. Preliminaries

Definition 2.1

Let X be the universal set and $\mathcal{P}(\mathrm{X})$ be the power set of X. A pair (F,E) is called a soft set over X, where E is the set of all parameters such that F is a mapping given by $\mathrm{F}:\mathrm{E}\to\mathcal{P}(\mathrm{X})$ **Definition 2.2**

Let X be a vector space over a field $\mathrm{K}(\mathrm{K}=\mathbb{R})$ and the parameter set E be the real number set $\mathbb{R}$. The soft set (F,E) is said to be a soft vector, if there is exactly one $e\in\mathrm{E}$, such that $\mathrm{F}(e)=\{x\}$ for some $x\in\mathrm{X}$ and $\mathrm{F}(e')=\phi,\ \forall\ e'\in E/\{e\}$, it is denoted by $\tilde{x}_e$ The set of all soft vectors over $\tilde{\mathrm{X}}$ is denoted by $\mathrm{SV}(\tilde{\mathrm{X}})$, the set $\mathrm{SV}(\tilde{\mathrm{X}})$ is called a soft vector space.

Definition 2.3

Let $\mathbb{R}$ be the set of all real numbers and $\mathrm{B}(\mathbb{R})$ be the collection of all non-empty bounded subsets of $\mathbb{R}$ and E taken as a set of parameters. Then a mapping $\mathrm{F}:\mathrm{E}\to\mathrm{B}(\mathbb{R})$ is called a soft real set. If a soft real set is a singleton soft set, it will be called a soft real number and denoted by $\tilde{r},\tilde{s},\tilde{t}$ etc .

Definition 2.4

Let $\mathrm{SV}(\tilde{\mathrm{X}})$ be a soft vector space over a field of soft real numbers. Then a mapping $\|\,.\,\|:\mathrm{SV}(\tilde{\mathrm{X}})\to\mathbb{R}^{+}(\mathrm{E})$ is said to be a soft norm on $\mathrm{SV}(\tilde{\mathrm{X}})$ if $\|\,.\,\|$ satisfies the following conditions

i) $\|\tilde{x}_e\| \tilde{\geq} \tilde{0}$ for all $SV(\tilde{X})$ and $\|\tilde{x}_e\| = \tilde{0} \Leftrightarrow \tilde{x}_e = \tilde{\theta}_0$

ii) $\|\tilde{r}.\tilde{x}_e\| = |\tilde{r}|\|\tilde{x}_e\|$ for all $\tilde{x}_e \tilde{\in} SV(\tilde{X})$ for every soft scalar $\tilde{r} \tilde{\in} \mathbb{R}^+(E)$

iii) $\|\tilde{x}_e + \tilde{y}_{e'}\| = \|\tilde{x}_e\| + \|\tilde{y}_{e'}\|$ for all $\tilde{x}_e, \tilde{y}_{e'} \tilde{\in} SV(\tilde{X})$

The soft vector space $SV(\tilde{X})$ with a soft norm $\|.\|$ on $\tilde{X}$ is said to be a soft normed linear space and is denoted by $(\tilde{X}, \|.\|)$.

Definition 2.5

A pair (F,A) is called fuzzy soft set over X if f is a mapping given by $f : A \to I^X$, $A \subseteq E$. So for every $e \in A$, $f(e)$ is a fuzzy subset of X with membership function $f_e : X \to [0,1]$. The fuzzy soft set is denoted by f_A and the set of all fuzzy soft sets is denoted by $\mathcal{FS}(X_E)$.

Definition 2.6

A fuzzy soft set f_E over X is called a fuzzy soft point if for the element $e^* \in E$

$$f_e(x) = \begin{cases} \lambda_x & if \quad e = e^* \\ 0 & otherwise \end{cases} \quad \text{for every } e \in E.$$

Otherwise, for the element $x^* \in X$, $f_e(x) = \begin{cases} \lambda & if \quad x = x^* \\ 0 & otherwise \end{cases}$ for every $x \in X$,

where $\lambda \in (0,1]$. The set of all fuzzy soft points is denoted as $\tilde{x}_E$.

3. Fuzzy soft continuity and boundedness using fuzzy soft normed linear space

Let $SSP(\tilde{X})$ or $\mathcal{S}(\tilde{X})$ be the set of all soft points on soft normed linear space $\tilde{X}$. The map $\tilde{X}_E$ from $SSP(\tilde{X})$ to I^X is called the fuzzy soft set on $SSP(\tilde{X})$ and the set of all fuzzy sets on $\tilde{X}$ is denoted as F $(SSP(\tilde{X}))$ or F S $(\tilde{X})$.

Definition 3.1

Let $\tilde{X}$ be an absolute soft linear space over the scalar field K. Suppose $*$ is a continuous t-norm, $\mathbb{R}(A^*)$ is the set of all non negative soft real numbers and $SSP(\tilde{X})$ denote the set of all soft points on $\tilde{X}$. A fuzzy subset Γ on $SSP(\tilde{X}) \times \mathbb{R}(A^*)$ is called a fuzzy soft norm on $\tilde{X}$ if and only if for $\tilde{x}_e, \tilde{y}_{e'} \tilde{\in} SSP(\tilde{X})$ and $\tilde{k} \tilde{\in} K$ (where $\tilde{k}$ is a soft scalar) the following conditions hold

1) $\Gamma(\tilde{x}_e, \tilde{t}) = 0 \ \forall \ \tilde{t} \tilde{\in} \mathbb{R}(A^*)$ with $\tilde{t} \tilde{\leq} \tilde{0}$
2) $\Gamma(\tilde{x}_e, \tilde{t}) = 1 \ \forall \ \tilde{t} \tilde{\in} \mathbb{R}(A^*)$ with $\tilde{t} \tilde{>} \tilde{0}$ if and only if $\tilde{x}_e = \tilde{\theta}_0$
3) $\Gamma(\tilde{k} \, \tilde{x}_e, \tilde{t}) = \Gamma\left(\tilde{x}_e, \frac{\tilde{t}}{|\tilde{k}|}\right)$ if $\tilde{k} \neq \tilde{0} \ \forall \ \tilde{t} \tilde{\in} \mathbb{R}(A^*)$, $\tilde{t} \tilde{>} \tilde{0}$
4) $\Gamma(\tilde{x}_e \oplus \tilde{y}_{e'}, \tilde{t} \oplus \tilde{s}) \tilde{\geq} \Gamma(\tilde{x}_e, \tilde{t}) * \Gamma(\tilde{y}_e, \tilde{s})$, $\forall \ \tilde{s}, \tilde{t} \tilde{\in} \mathbb{R}(A^*)$, $\tilde{x}_e, \tilde{y}_{e'} \tilde{\in} SSP(\tilde{X})$
5) $\Gamma(\tilde{x}_e, .)$ is a continuous nondecreasing function of $\mathbb{R}(A^*)$ and $\lim_{\tilde{t} \to \infty} \Gamma(\tilde{x}_e, \tilde{t}) = 1$

The triplet $(\tilde{X}, \Gamma, *)$ will be referred to as a fuzzy soft normed linear space.

Definition 3.2

Let $S(\tilde{X})$ and $S(\tilde{Y})$ be set of all soft points on soft normed linear space $\tilde{X}$ and $\tilde{Y}$ respectively, let E and E' be the corresponding parameter sets. The map $T: S(\tilde{X}) \to S(\tilde{Y})$ takes the soft point $\tilde{x}_e$ on $\tilde{X}$ to the soft point $T(\tilde{x}_e)$ on $\tilde{Y}$.

Definition 3.3

Let $S(\tilde{X})$ and $S(\tilde{Y})$ be the set of all soft points on $\tilde{X}$ and $\tilde{Y}$, let $FS(\tilde{X})$ and $FS(\tilde{Y})$ be the set of all fuzzy sets on $S(\tilde{X})$ and $S(\tilde{Y})$ respectively. The map $T_{up}: FS(\tilde{X}) \to FS(\tilde{Y})$ is said to be a fuzzy soft map which maps the fuzzy soft set $\tilde{X}_E$ on $S(\tilde{X})$ to the fuzzy soft set $T_{up}(\tilde{X}_E)$ on $S(\tilde{Y})$ defined by

$$\left[T_{up}(\tilde{X}_E)\right](\tilde{y}_{e'_1}) = \begin{cases} \sup\limits_{x=u^{-1}(y)} \left[\sup\limits_{e_1 = p^{-1}(e'_1)} \tilde{X}_E(e_1) \right](x) & \text{if } p^{-1}(e'_1) \neq \phi \text{ and } u^{-1}(y) \neq \phi \\ 0 & \text{otherwise} \end{cases}$$

for all $x \in X$ and $e \in E$, where $u: X \to Y$ and $p: E \to E'$ are ordinary functions.

Definition 3.4

Let $T_{up}: FS(\tilde{X}) \to FS(\tilde{X})$ be a linear operator, where $\tilde{X}$ is fuzzy soft normed linear space. The operator T_{up} is called bounded if there exists some positive fuzzy soft real number $\tilde{M}$ such that for all $\tilde{X}_E \tilde{\in} FS(\tilde{X})$, $\left\| T_{up}(\tilde{X}_E) \right\| \tilde{\leq} \tilde{M} \left\| \tilde{X}_E \right\|$.

Definition 3.5

A fuzzy soft linear operator $T_{up}: FS(\tilde{X}) \to FS(\tilde{X})$ satisfying the condition for every $\alpha \in (0,1]$ and $\tilde{x}_w \tilde{\in} S(\tilde{X})$, $\left\{ \left(T_{up}(\tilde{X}_E)\right)(\tilde{x}_w) : \tilde{X}_E \tilde{\in} FS(\tilde{X}) \text{ such that } (\tilde{X}_E)(\tilde{x}_w) = \alpha \right\}$ is a singleton set. Then for each $\tilde{x}_w \tilde{\in} S(\tilde{X})$, define the mapping $(T_{up})_{\tilde{x}_w} : [0,1] \to [0,1]$ by $(T_{up})_{\tilde{x}_w}(\alpha) = T_{up}(\tilde{X}_E)(\tilde{x}_w)$, for all $\alpha \in [0,1]$ and $\tilde{x}_w \tilde{\in} SSP(\tilde{X})$ such that $\tilde{X}_E(\tilde{x}_w) = \alpha$ is a fuzzy soft linear operator.

Definition 3.6

Let T_{up} be a bounded fuzzy soft linear operator from $FS(\tilde{X})$ into $FS(\tilde{X})$. The norm of the operator T_{up} denoted by $\left\| T_{up} \right\|$, is a fuzzy soft real number defined as

$$\left\| T_{up} \right\|(\tilde{x}_w) = \inf \left\{ \tilde{t} \in \mathbb{R}; \left\| T_{up}(\tilde{X}_E) \right\|(\tilde{x}_w) \tilde{\leq} \tilde{t} \left\| \tilde{X}_E \right\| \left\| \tilde{x}_w \right\| \text{ for each } \tilde{X}_E \tilde{\in} FS(\tilde{X}) \right\} \text{ for every } \tilde{x}_w \tilde{\in} S(\tilde{X})$$

Definition 3.7

Let $\tilde{X}$ be soft normed linear space. Define for each $\tilde{x}_w \tilde{\in} S(\tilde{X})$, $\left\| T_{up} \right\|(\tilde{x}_w) = \left\| (T_{up})_{\tilde{x}_w} \right\|$ where $\left\| (T_{up})_{\tilde{x}_w} \right\|$ is the norm of the fuzzy soft linear operator $(T_{up})_{\tilde{x}_w} : [0,1] \to [0,1]$.

Definition 3.8

Let $\tilde{X}$ be soft normed linear space and $T_{up}: FS(\tilde{X}) \to FS(\tilde{X})$ be a fuzzy soft linear operator. Then

i) $\left\| T_{up} \right\|(\tilde{x}_w) = \sup \left\{ \left\| T_{up}(\tilde{X}_E) \right\|(\tilde{x}_w) : \left\| \tilde{X}_E \right\| \tilde{\leq} \tilde{1} \right\} = \left\| (T_{up})_{\tilde{x}_w} \right\|$ for each $\tilde{x}_w \tilde{\in} S(\tilde{X})$

ii) $\left\| T_{up} \right\|(\tilde{x}_w) = \sup \left\{ \left\| T_{up}(\tilde{X}_E) \right\|(\tilde{x}_w) : \left\| \tilde{X}_E \right\| = \tilde{1} \right\} = \left\| (T_{up})_{\tilde{x}_w} \right\|$ for each $\tilde{x}_w \tilde{\in} S(\tilde{X})$

iii) $\left\| T_{up} \right\|(\tilde{x}_w) = \sup \left\{ \dfrac{\left\| T_{up}(\tilde{X}_E) \right\|}{\left\| \tilde{X}_E \right\|}(\tilde{x}_w) : \left\| \tilde{X}_E \right\| \neq \tilde{0}, \textit{ for all } \tilde{x}_w \tilde{\in} S(\tilde{X}) \right\} = \left\| (T_{up})_{\tilde{x}_w} \right\|$ for each $\tilde{x}_w \tilde{\in} S(\tilde{X})$

Definition 3.9

The fuzzy soft operator $\mathrm{T}_{up}:\mathrm{FS}\left(\tilde{\mathrm{X}}\right)\rightarrow\mathrm{FS}\left(\tilde{\mathrm{Y}}\right)$ is said to be continuous at $\left(\tilde{\mathrm{X}}_{\mathrm{E}}\right)_0$ if for every sequence $\left\{\left(\tilde{\mathrm{X}}_{\mathrm{E}}\right)_n\right\}$ of fuzzy soft sets on $\tilde{\mathrm{X}}$ converges to $\left(\tilde{\mathrm{X}}_{\mathrm{E}}\right)_0$ then $\mathrm{T}_{up}\left(\tilde{\mathrm{X}}_{\mathrm{E}}\right)_n$ converges to $\mathrm{T}_{up}\left(\tilde{\mathrm{X}}_{\mathrm{E}}\right)_0$ as $n\rightarrow\infty$. Otherwise, if for every sequence $\left\{\left(\tilde{\mathrm{X}}_{\mathrm{E}}\right)_n\right\}$ of fuzzy soft sets with $\left\|\left(\tilde{\mathrm{X}}_{\mathrm{E}}\right)_n\left(\tilde{y}_{e_n}\right)-\left(\tilde{\mathrm{X}}_{\mathrm{E}}\right)_0\left(\tilde{y}_{e_0}\right)\right\|\tilde{<}\delta$ implies $\left\|\mathrm{T}_{up}\left(\tilde{\mathrm{X}}_{\mathrm{E}}\right)_n\left(\tilde{y}_{e'_n}\right)-\mathrm{T}_{up}\left(\tilde{\mathrm{X}}_{\mathrm{E}}\right)_0\left(\tilde{y}_{e'_0}\right)\right\|\tilde{<}\in$ where $\delta>0$ and $\in>0$.

If T_{up} is fuzzy soft continuous at each fuzzy soft set of $\tilde{\mathrm{X}}$ then T_{up} is said to be fuzzy soft continuous operator.

The fuzzy soft linear operator $\left(\mathrm{T}_{up}\right)_{\tilde{x}_w}$ is continuous on $[0,1]$ for each $\tilde{x}_w\tilde{\in}\mathrm{S}\left(\tilde{\mathrm{X}}\right)$ if T_{up} is a continuous fuzzy soft linear operator.

Definition 3.10

Let $\tilde{\mathrm{X}}$ be a soft normed linear space over a finite set of parameters A. Let $\left\{\left(\mathrm{T}_{up}\right)_{\tilde{x}_w}:\tilde{x}_w\tilde{\in}\mathrm{S}\left(\tilde{\mathrm{X}}\right)\right\}$ be a family of continuous linear operators such that $\left(\mathrm{T}_{up}\right)_{\tilde{x}_w}:[0,1]\rightarrow[0,1]$ for each $\tilde{x}_w$. Then the operator $\mathrm{T}_{up}:\mathrm{FS}\left(\tilde{\mathrm{X}}\right)\rightarrow\mathrm{FS}\left(\tilde{\mathrm{X}}\right)$ defined by $\mathrm{T}_{up}\left(\tilde{x}_w\right)=\left(\mathrm{T}_{up}\right)_{\tilde{x}_w}$, $\forall\ \tilde{x}_w\tilde{\in}\mathrm{S}\left(\tilde{\mathrm{X}}\right)$ is a continuous fuzzy soft linear operator.

Definition 3.11

The continuous fuzzy soft linear transformation from $\mathrm{FS}\left(\tilde{\mathrm{X}}\right)$ to $\mathbb{R}_{\mathrm{E}'}$ according as $\mathbb{R}_{\mathrm{E}'}$ is a set of all fuzzy soft real number is called continuous fuzzy soft linear functional.

The functional $\mathrm{T}_{up}:\mathrm{FS}\left(\tilde{\mathrm{X}}\right)\rightarrow\mathbb{R}_{\mathrm{E}'}$ is defined as

$$\left[\mathrm{T}_{up}\left(\tilde{\mathrm{X}}_{\mathrm{E}}\right)\right]_{e'}(\tilde{r})=\begin{cases}\sup\limits_{x=u^{-1}(y)}\left[\sup\limits_{e=p^{-1}(e')}\tilde{\mathrm{X}}_{\mathrm{E}}(e)\right](x) & \text{if } p^{-1}(e')\neq\phi \text{ and } u^{-1}(y)\neq\phi\\ 0 & \text{otherwise}\end{cases}$$

Definition 3.12

Let $\mathrm{FS}\left(\tilde{\mathrm{X}}\right)$ be the set of all fuzzy soft sets on soft normed linear space $\tilde{\mathrm{X}}$. For each $\tilde{x}_w\tilde{\in}\mathrm{S}\left(\tilde{\mathrm{X}}\right)$, the set $\mathrm{W}([0,1])$ of all continuous linear operators from $[0,1]$ to $[0,1]$ is a vector space with respect to the usual operations of addition and scalar multiplication defined as

1) $\left(\mathrm{S}_{up}\right)_{\tilde{x}_w}+\left(\mathrm{T}_{up}\right)_{\tilde{x}_w}=\left(\mathrm{T}_{up}\right)_{\tilde{x}_w}+\left(\mathrm{S}_{up}\right)_{\tilde{x}_w}$

$$=\left(\mathrm{S}_{up}+\mathrm{T}_{up}\right)\left(\tilde{x}_w\right)$$

$$\left(\mathrm{S}_{up}\right)_{\tilde{x}_w}+\left(\mathrm{T}_{up}\right)_{\tilde{x}_w}=\left(\mathrm{S}_{up}+\mathrm{T}_{up}\right)_{\tilde{x}_w}$$

2) $\tilde{r}_{\tilde{x}_w}\left(\mathrm{T}_{up}\right)_{\tilde{x}_w}=\tilde{r}\left(\tilde{x}_w\right).\mathrm{T}_{up}\left(\tilde{x}_w\right)$

$$=\left(\tilde{r}.\mathrm{T}_{up}\right)\left(\tilde{x}_w\right)$$

$$=\left(\tilde{r}.\mathrm{T}_{up}\right)_{\tilde{x}_w}$$

3) $0_{\tilde{x}_w}$ is the identity

4) $-\mathrm{T}_{up}$ is the inverse of T_{up}.

5)

4. Hahn Banach Theorem

Theorem 4.1

Let $\tilde{X}$ be soft normed linear space. Let $T_{up} : FS(\tilde{X}) \to FS(\tilde{X})$ be a bounded fuzzy soft linear operator. Then $(T_{up})_{\tilde{x}_w}$ is bounded on $[0,1]$ for each $\tilde{x}_w \tilde{\in} S(\tilde{X})$.

Proof

Let $\tilde{x}_w \tilde{\in} S(\tilde{X})$ and assume T_{up} is a bounded fuzzy soft linear operator then there exists a positive fuzzy soft real number $\tilde{M}$ such that for all $\tilde{X}_E \tilde{\in} FS(\tilde{X})$,

$$\left\|T_{up}(\tilde{X}_E)\right\| \tilde{\leq} \tilde{M}\left\|\tilde{X}_E\right\|$$

$$\text{i.e., } \left\|T_{up}(\tilde{X}_E)\right\|(\tilde{x}_w) \tilde{\leq} \left(\tilde{M}\left\|\tilde{X}_E\right\|\right)(\tilde{x}_w)$$

$$\text{i.e., } \left\|(T_{up})_{\tilde{x}_w}(\tilde{X}_E)(\tilde{x}_w)\right\| \tilde{\leq} \tilde{M}_{\tilde{x}_w}\left\|\tilde{X}_E\right\|(\tilde{x}_w), \quad \forall \quad \tilde{X}_E \tilde{\in} FS(\tilde{X})$$

$$\text{i.e., } \left\|(T_{up})_{\tilde{x}_w}(\tilde{X}_E)(\tilde{x}_w)\right\| \tilde{\leq} \tilde{M}_{\tilde{x}_w}\left\|\tilde{X}_E(\tilde{x}_w)\right\|$$

$$\Rightarrow \left\|(T_{up})_{\tilde{x}_w}(\alpha)\right\| \tilde{\leq} \tilde{M}_{\tilde{x}_w}\left\|\alpha\right\|, \quad \forall \quad \alpha \tilde{\in} [0,1]$$

This shows that $(T_{up})_{\tilde{x}_w}$ is bounded on $[0,1]$ since $\tilde{x}_w$ is arbitrary .

Theorem 4.2

Let $\tilde{X}$ be a soft normed linear space and $\left\{(T_{up})_{\tilde{x}_w} : \tilde{x}_w \tilde{\in} S(\tilde{X})\right\}$ be a family of bounded linear operators such that $(T_{up})_{\tilde{x}_w} : [0,1] \to [0,1]$. Then the fuzzy soft linear operator $T_{up} : FS(\tilde{X}) \to FS(\tilde{X})$ defined by $\left(T_{up}(\tilde{X}_E)\right)(\tilde{x}_w) = (T_{up})_{\tilde{x}_w}(\tilde{X}_E)(\tilde{x}_w)$, $\forall\ \tilde{x}_w \tilde{\in} S(\tilde{X})$ is a bounded fuzzy soft linear operator.

Proof

Given $T_{up} : FS(\tilde{X}) \to FS(\tilde{X})$ is a fuzzy soft linear operator and if $(T_{up})_{\tilde{x}_w}$ is a bounded linear operator there exists a positive real number $\tilde{M}_{\tilde{x}_w}$ such that for all $\alpha \tilde{\in} [0,1]$, $\left\|(T_{up})_{\tilde{x}_w}(\alpha)\right\| \tilde{\leq} \tilde{M}_{\tilde{x}_w}\left\|\alpha\right\|$

Consider a positive fuzzy soft real number $\tilde{M}$ such that $\tilde{M} = \tilde{M}_{\tilde{x}_w}$, $\forall\ \tilde{x}_w \tilde{\in} S(\tilde{X})$.

$$\left\|T_{up}(\tilde{X}_E)\right\|(\tilde{x}_w) = \left\|(T_{up})_{\tilde{x}_w}(\tilde{X}_E)(\tilde{x}_w)\right\| \text{ for all } \alpha \tilde{\in} [0,1],$$

$$\Rightarrow \left\|T_{up}(\tilde{X}_E)\right\|(\tilde{x}_w) = \left\|(T_{up})_{\tilde{x}_w}(\alpha)\right\|$$

$$\Rightarrow \left\|T_{up}(\tilde{X}_E)\right\|(\tilde{x}_w) \tilde{\leq} \tilde{M}_{\tilde{x}_w}\left\|\alpha\right\|$$

$$= \tilde{M}_{\tilde{x}_w}\left\|\tilde{X}_E(\tilde{x}_w)\right\|$$

This implies, $\left\|T_{up}(\tilde{X}_E)\right\|(\tilde{x}_w) \tilde{\leq} \tilde{M}\left\|\tilde{X}_E\right\|(\tilde{x}_w)$

That is for all $\tilde{X}_E \tilde{\in} FS(\tilde{X})$, $\left\|T_{up}(\tilde{X}_E)\right\| = \tilde{M}\left\|\tilde{X}_E\right\|$

Therefore, the fuzzy soft linear operator T_{up} is bounded.

Theorem4.3

Let $\tilde{X}$ be a soft normed linear space. Let $\left\{\left(T_{up}\right)_{\tilde{x}_w} : \tilde{x}_w \tilde{\in} S\left(\tilde{X}\right)\right\}$ be a family of continuous linear operators such that $\left(T_{up}\right)_{\tilde{x}_w} : [0,1] \to [0,1]$ for each $\tilde{x}_w$. Then the fuzzy soft linear operator $T_{up} : FS\left(\tilde{X}\right) \to FS\left(\tilde{X}\right)$ defined by $\left(T_{up}\left(\tilde{X}_E\right)\right)\left(\tilde{x}_w\right) = \left(T_{up}\right)_{\tilde{x}_w}\left(\tilde{X}_E\right)\left(\tilde{x}_w\right)$, $\forall\ \tilde{x}_w \tilde{\in} S\left(\tilde{X}\right)$ is a continuous fuzzy soft linear operator.

Proof

Given $T_{up} : FS\left(\tilde{X}\right) \to FS\left(\tilde{X}\right)$ is a fuzzy soft linear operator defined as $\left(T_{up}\left(\tilde{X}_E\right)\right)\left(\tilde{x}_w\right) = \left(T_{up}\right)_{\tilde{x}_w}\left(\tilde{X}_E\right)\left(\tilde{x}_w\right)$

For each $\tilde{x}_w \tilde{\in} SSP\left(\tilde{X}\right)$, the linear operator $\left(T_{up}\right)_{\tilde{x}_w}$ is continuous, hence it is bounded.

Therefore T_{up} is a bounded fuzzy soft linear operator $\forall\ \tilde{x}_w \tilde{\in} S\left(\tilde{X}\right)$.

This implies T_{up} is continuous fuzzy soft linear operator.

Theorem 4.4

Let $\tilde{X}$ be the soft normed linear space, the space $W\left(FS\left(\tilde{X}\right)\right)$ of all continuous fuzzy soft linear operators on $FS\left(\tilde{X}\right)$ is itself a fuzzy soft normed linear space with respect to point wise linear operations and the norm is defined by

$$\left\|\left(T_{up}\right)_{\tilde{x}_w}\right\| = \left\|T_{up}\right\|\left(\tilde{x}_w\right)$$

$$= \inf\left\{\tilde{t} \tilde{\in} \mathbb{R} ; \left\|T_{up}\left(\tilde{X}_E\right)\right\|\left(\tilde{x}_w\right) \tilde{\leq} \tilde{t}\left\|\tilde{X}_E\right\|\left\|\tilde{x}_w\right\|, \textit{ for each } \tilde{X}_E \tilde{\in} FS\left(\tilde{X}\right)\right\}$$

Further if $FS\left(\tilde{X}\right)$ is a fuzzy soft Banach space then $W\left(FS\left(\tilde{X}\right)\right)$ is also a Banach space.

Proof

Given $W\left(FS\left(\tilde{X}\right)\right)$ is a set of all continuous fuzzy soft linear operators.

Therefore if $T_{up}, S_{up} \tilde{\in} W\left(FS\left(\tilde{X}\right)\right)$ then $\left\|T_{up}\left(\tilde{X}_E\right)\right\| \tilde{\leq} \tilde{M}\left\|\tilde{X}_E\right\|$ and $\left\|S_{up}\left(\tilde{X}_E\right)\right\| \tilde{\leq} \tilde{M}\left\|\tilde{X}_E\right\|$

From Theorem 4.1

$$\left\|\left(T_{up}\right)_{\tilde{x}_w}(\alpha)\right\| \tilde{\leq} \tilde{M}_{\tilde{x}_w}\|\alpha\| \text{ and } \left\|\left(S_{up}\right)_{\tilde{x}_w}(\alpha)\right\| \tilde{\leq} \tilde{M}_{\tilde{x}_w}\|\alpha\|$$

Cleary $W\left(\left(0,1\right]\right)$ is a linear space

1) $$\left\|\left(T_{up} + S_{up}\right)_{\tilde{x}_w}(\alpha)\right\| = \left\|\left[\left(T_{up}\right)_{\tilde{x}_w} + \left(S_{up}\right)_{\tilde{x}_w}\right](\alpha)\right\|$$

$$= \left\|\left(T_{up}\right)_{\tilde{x}_w}(\alpha) + \left(S_{up}\right)_{\tilde{x}_w}(\alpha)\right\|$$

$$\tilde{\leq} \tilde{M}_{\tilde{x}_w}\|\alpha\| + \tilde{M}_{\tilde{x}_w}\|\alpha\|$$

$$= 2\tilde{M}_{\tilde{x}_w}\|\alpha\|$$

$$\left\|\left(T_{up} + S_{up}\right)_{\tilde{x}_w}(\alpha)\right\| = 2\tilde{M}_{\tilde{x}_w}\|\alpha\|$$

This implies, $\left(T_{up} + S_{up}\right)_{\tilde{x}_w} \tilde{\in} W\left([0,1]\right)$.

2) Let $\tilde{r}$ be any scalar and $\left(T_{up}\right)_{\tilde{x}_w} \tilde{\in} W\left([0,1]\right)$

$$\left\|\left(\tilde{r}.T_{up}\right)_{\tilde{x}_w}(\alpha)\right\| = \left\|\tilde{r}_{\tilde{x}_w}.\left(T_{up}\right)_{\tilde{x}_w}(\alpha)\right\|$$

$$= \tilde{r}_{\tilde{x}_w} \left\| \left(\mathrm{T}_{up}\right)_{\tilde{x}_w}(\alpha) \right\|$$
$$\tilde{\le} \tilde{r}_{\tilde{x}_w}.\mathrm{M}_{\tilde{x}_w}\|\alpha\|$$
$$\left\| \left(\tilde{r}.\mathrm{T}_{up}\right)_{\tilde{x}_w}(\alpha) \right\| \tilde{\le} \tilde{r}_{\tilde{x}_w}.\mathrm{M}_{\tilde{x}_w}\|\alpha\|$$

This implies $\left(\tilde{r}.\mathrm{T}_{up}\right)_{\tilde{x}_w} \tilde{\in} \mathrm{W}([0,1])$.

3) $\left(0_{up}\right)_{\tilde{x}_w} : [0,1] \to [0,1]$ such that $\left(0_{up}\right)_{\tilde{x}_w}(\alpha) = 0$, for all $0 \tilde{\in} [0,1]$.

4) $\left(\text{-}\mathrm{T}_{up}\right)_{\tilde{x}_w} \tilde{\in} \mathrm{W}([0,1])$

Consider $\left(\mathrm{T}_{up}\right)_{\tilde{x}_w} \tilde{\in} \mathrm{W}([0,1])$.

Since,

1) $\left\| \mathrm{T}_{up}\left(\tilde{\mathrm{X}}_{\mathrm{E}}\right) \right\| \quad \tilde{\ge} \tilde{0}$,

$$\left\| \mathrm{T}_{up}\left(\tilde{\mathrm{X}}_{\mathrm{E}}\right) \right\| \left(\tilde{x}_w\right) \ \tilde{\ge} \tilde{0}$$
$$\left\| \left(\mathrm{T}_{up}\right)_{\tilde{x}_w}\left(\tilde{\mathrm{X}}_{\mathrm{E}}\right)\left(\tilde{x}_w\right) \right\| \tilde{\ge} \tilde{0}$$
$$\left\| \left(\mathrm{T}_{up}\right)_{\tilde{x}_w}(\alpha) \right\| \quad \tilde{\ge} \tilde{0} \text{ where } \tilde{\mathrm{X}}_{\mathrm{E}}\left(\tilde{x}_w\right) = \alpha \text{ for all } \alpha \tilde{\in} [0,1].$$

If $\left\| \left(\mathrm{T}_{up}\right)_{\tilde{x}_w} \right\| = \tilde{0}$ then $\left\| \left(\mathrm{T}_{up}\right)_{\tilde{x}_w}(\alpha) \right\| = \tilde{0}$.

$\left(\mathrm{T}_{up}\right)_{\tilde{x}_w}(\alpha) = \tilde{0}$, for all $\alpha \tilde{\in} [0,1]$

This implies, $\left(\mathrm{T}_{up}\right)_{\tilde{x}_w} = \tilde{0}$

2) Consider $\left(\mathrm{T}_{up}\right)_{\tilde{x}_w}, \left(\mathrm{S}_{up}\right)_{\tilde{x}_w} \tilde{\in} \mathrm{W}([0,1])$

$$\left\| \left(\mathrm{T}_{up}\right)_{\tilde{x}_w} + \left(\mathrm{S}_{up}\right)_{\tilde{x}_w} \right\| = \left\| \left(\mathrm{T}_{up} + \mathrm{S}_{up}\right)_{\tilde{x}_w} \right\|$$
$$= \inf\left\{ \tilde{t} + \tilde{s} \tilde{\in} \mathbb{R}; \left\| \left(\mathrm{T}_{up} + \mathrm{S}_{up}\right)\left(\tilde{\mathrm{X}}_{\mathrm{E}}\right) \right\| \left(\tilde{x}_w\right) \tilde{\le} \left(\tilde{t} + \tilde{s}\right) \left\|\tilde{\mathrm{X}}_{\mathrm{E}}\right\| \left\|\tilde{x}_w\right\|, \textit{ for each } \tilde{\mathrm{X}}_{\mathrm{E}} \tilde{\in} \mathrm{F\,S}\left(\tilde{\mathrm{X}}\right) \right\}$$
$$\tilde{\le} \inf\left\{ \tilde{t} + \tilde{s} \tilde{\in} \mathbb{R}; \left\| \mathrm{T}_{up}\left(\tilde{\mathrm{X}}_{\mathrm{E}}\right) + \mathrm{S}_{up}\left(\tilde{\mathrm{X}}_{\mathrm{E}}\right) \right\| \left(\tilde{x}_w\right) \tilde{\le} \left(\tilde{t} + \tilde{s}\right) \left\|\tilde{\mathrm{X}}_{\mathrm{E}}\right\| \left\|\tilde{x}_w\right\|, \textit{ for each } \tilde{\mathrm{X}}_{\mathrm{E}} \tilde{\in} \mathrm{F\,S}\left(\tilde{\mathrm{X}}\right) \right\}$$
$$\left\| \left(\mathrm{T}_{up}\right)_{\tilde{x}_w} + \left(\mathrm{S}_{up}\right)_{\tilde{x}_w} \right\| \tilde{\le} \inf\left\{ \tilde{t} \tilde{\in} \mathbb{R}; \left\| \mathrm{T}_{up}\left(\tilde{\mathrm{X}}_{\mathrm{E}}\right) \right\| \left(\tilde{x}_w\right) \tilde{\le} \tilde{t} \left\|\tilde{\mathrm{X}}_{\mathrm{E}}\right\| \left\|\tilde{x}_w\right\|, \textit{ for each } \tilde{\mathrm{X}}_{\mathrm{E}} \tilde{\in} \mathrm{F\,S}\left(\tilde{\mathrm{X}}\right) \right\}$$
$$+ \inf\left\{ \tilde{s} \tilde{\in} \mathbb{R}; \left\| \mathrm{S}_{up}\left(\tilde{\mathrm{X}}_{\mathrm{E}}\right) \right\| \left(\tilde{x}_w\right) \tilde{\le} \tilde{s} \left\|\tilde{\mathrm{X}}_{\mathrm{E}}\right\| \left\|\tilde{x}_w\right\|, \textit{ for each } \tilde{\mathrm{X}}_{\mathrm{E}} \tilde{\in} \mathrm{F\,S}\left(\tilde{\mathrm{X}}\right) \right\}$$
$$\left\| \left(\mathrm{T}_{up}\right)_{\tilde{x}_w} + \left(\mathrm{S}_{up}\right)_{\tilde{x}_w} \right\| \tilde{\le} \left\| \left(\mathrm{T}_{up}\right)_{\tilde{x}_w} \right\| + \left\| \left(\mathrm{S}_{up}\right)_{\tilde{x}_w} \right\|$$

3) $\left\| \tilde{r}_{\tilde{x}_w}.\left(\mathrm{T}_{up}\right)_{\tilde{x}_w} \right\| = \left\| \left(\tilde{r}.\mathrm{T}_{up}\right)_{\tilde{x}_w} \right\|$

$$= \inf\left\{ \tilde{r}_{\tilde{x}_w}.\tilde{t} \tilde{\in} \mathbb{R}; \ \left\| \tilde{r}.\mathrm{T}_{up}\left(\tilde{\mathrm{X}}_{\mathrm{E}}\right) \right\| \left(\tilde{x}_w\right) \tilde{\le} \tilde{r}_{\tilde{x}_w}.\,\tilde{t} \left\|\tilde{\mathrm{X}}_{\mathrm{E}}\right\| \left\|\tilde{x}_w\right\|, \textit{ for each } \tilde{\mathrm{X}}_{\mathrm{E}} \tilde{\in} \mathrm{F\,S}\left(\tilde{\mathrm{X}}\right) \right\}$$
$$= \inf\left\{ \tilde{r}_{\tilde{x}_w}.\,\tilde{t} \tilde{\in} \mathbb{R}; \ \tilde{r}_{\tilde{x}_w} \left\| \mathrm{T}_{up}\left(\tilde{\mathrm{X}}_{\mathrm{E}}\right) \right\| \left(\tilde{x}_w\right) \tilde{\le} \tilde{r}_{\tilde{x}_w}.\tilde{t} \left\|\tilde{\mathrm{X}}_{\mathrm{E}}\right\| \left\|\tilde{x}_w\right\|, \textit{ for each } \tilde{\mathrm{X}}_{\mathrm{E}} \tilde{\in} \mathrm{F\,S}\left(\tilde{\mathrm{X}}\right) \right\}$$
$$= \inf\left\{ \tilde{r}_{\tilde{x}_w}.\,\tilde{t} \tilde{\in} \mathbb{R}; \ \left\| \mathrm{T}_{up}\left(\tilde{\mathrm{X}}_{\mathrm{E}}\right) \right\| \left(\tilde{x}_w\right) \tilde{\le} \tilde{t} \left\|\tilde{\mathrm{X}}_{\mathrm{E}}\right\| \left\|\tilde{x}_w\right\|, \textit{ for each } \tilde{\mathrm{X}}_{\mathrm{E}} \tilde{\in} \mathrm{F\,S}\left(\tilde{\mathrm{X}}\right) \right\}$$

$$\left\| \tilde{r}_{\tilde{x}_w} . \left(\mathrm{T}_{up}\right)_{\tilde{x}_w} \right\| = \tilde{r}_{\tilde{x}_w} . \left\| \left(\mathrm{T}_{up}\right)_{\tilde{x}_w} \right\|$$

Therefore $\mathrm{W}([0,1])$ is a normed linear space.

Consider a Cauchy sequence $\left\{ \left(\mathrm{T}_{up}\right)_{\tilde{x}_w} \right\}_n$ in $\mathrm{W}([0,1])$.

Then for a given $\epsilon > 0$, there exists a positive integer n_0 such that

$$\left\| \left(\left(\mathrm{T}_{up}\right)_{\tilde{x}_w} \right)_n - \left(\left(\mathrm{T}_{up}\right)_{\tilde{x}_w} \right)_m \right\| < \epsilon, \text{ for all } n, m > n_0$$

$$\begin{aligned} \left\| \left(\left(\mathrm{T}_{up}\right)_{\tilde{x}_w} \right)_n - \left(\left(\mathrm{T}_{up}\right)_{\tilde{x}_w} \right)_m \right\| & \\ &= \inf \left\{ \tilde{t} \,\tilde{\in}\, \mathbb{R}; \left\| \left[\left(\mathrm{T}_{up}\right)_n - \left(\mathrm{T}_{up}\right)_m \right] \left(\tilde{\mathrm{X}}_{\mathrm{E}}\right) \right\| (\tilde{x}_w) \,\tilde{\leq}\, \tilde{t} \left\| \tilde{\mathrm{X}}_{\mathrm{E}} \right\| \left\| \tilde{x}_w \right\|, \textit{ for each } \tilde{\mathrm{X}}_{\mathrm{E}} \,\tilde{\in}\, \mathrm{F\,S}\left(\tilde{\mathrm{X}}\right) \right\} \\ &< \epsilon \end{aligned}$$

This implies

$$\begin{aligned} \left\| \left[\left(\mathrm{T}_{up}\right)_n - \left(\mathrm{T}_{up}\right)_m \right] \left(\tilde{\mathrm{X}}_{\mathrm{E}}\right) \right\| (\tilde{x}_w) &\,\tilde{\leq}\, \epsilon \\ \left\| \left[\left(\mathrm{T}_{up}\right)_n - \left(\mathrm{T}_{up}\right)_m \right] \left(\tilde{\mathrm{X}}_{\mathrm{E}}\right) \right\| &\,\tilde{\leq}\, \epsilon \\ \left\| \left(\mathrm{T}_{up}\right)_n \left(\tilde{\mathrm{X}}_{\mathrm{E}}\right) - \left(\mathrm{T}_{up}\right)_m \left(\tilde{\mathrm{X}}_{\mathrm{E}}\right) \right\| &\,\tilde{\leq}\, \epsilon \end{aligned}$$

Since $\left\{ \left(\mathrm{T}_{up}\right)_n \right\}$ is a Cauchy sequence in $\mathrm{F\,S}\left(\tilde{\mathrm{X}}\right)$ and $\mathrm{F\,S}\left(\tilde{\mathrm{X}}\right)$ is a Banach space,

$$\left(\mathrm{T}_{up}\right)_n \left(\tilde{\mathrm{X}}_{\mathrm{E}}\right) \rightarrow \mathrm{T}_{up} \left(\tilde{\mathrm{X}}_{\mathrm{E}}\right)$$

To prove that $\left(\mathrm{T}_{up}\right)_{\tilde{x}_w}$ is continuous and $\left(\left(\mathrm{T}_{up}\right)_{\tilde{x}_w} \right)_n \rightarrow \left(\mathrm{T}_{up}\right)_{\tilde{x}_w}$ with respect to the norm of $\mathrm{W}\left(\mathrm{F\,S}\left(\tilde{\mathrm{X}}\right)\right)$ consider

$$\begin{aligned} \left\| \left(\mathrm{T}_{up}\right)_{\tilde{x}_w} (\alpha) \right\| &= \left\| \lim_{n \to \infty} \left(\left(\mathrm{T}_{up}\right)_{\tilde{x}_w} \right)_n (\alpha) \right\| \\ &= \lim_{n \to \infty} \left\| \left(\left(\mathrm{T}_{up}\right)_{\tilde{x}_w} \right)_n (\alpha) \right\| \\ &\,\tilde{\leq}\, \lim_{n \to \infty} \left\| \left(\left(\mathrm{T}_{up}\right)_{\tilde{x}_w} \right)_n \right\| \left\| \alpha \right\| \\ &\,\tilde{\leq}\, \sup_{\left\| \tilde{\mathrm{X}}_{\mathrm{E}} \right\| \tilde{\leq} 1} \left\{ \left\| \left(\mathrm{T}_{up}\right)_n \left(\tilde{\mathrm{X}}_{\mathrm{E}}\right) \right\| (\tilde{x}_w) \right\} \left\| \alpha \right\| \\ &\,\tilde{\leq}\, \sup_{\left\| \tilde{\mathrm{X}}_{\mathrm{E}} \right\| \tilde{\leq} 1} \left\{ \tilde{\mathrm{M}} . \left\| \tilde{\mathrm{X}}_{\mathrm{E}} \right\| (\tilde{x}_w) \right\} \left\| \alpha \right\| \\ &= \sup_{\left\| \tilde{\mathrm{X}}_{\mathrm{E}} \right\| \tilde{\leq} 1} \tilde{\mathrm{M}}_{\tilde{x}_w} . \left\| \tilde{\mathrm{X}}_{\mathrm{E}} \right\| (\tilde{x}_w) . \left\| \alpha \right\| \\ &\,\tilde{\leq}\, \sup_{\left\| \tilde{\mathrm{X}}_{\mathrm{E}} \right\| \tilde{\leq} 1} \tilde{\mathrm{M}}_{\tilde{x}_w} . \left\| \tilde{\mathrm{X}}_{\mathrm{E}} \right\| . \left\| \alpha \right\| \\ &\,\tilde{\leq}\, \tilde{\mathrm{M}}_{\tilde{x}_w} \left\| \alpha \right\| \end{aligned}$$

$$\left\| \left(\mathrm{T}_{up}\right)_{\tilde{x}_w} (\alpha) \right\| \,\tilde{\leq}\, \tilde{\mathrm{M}}_{\tilde{x}_w} \left\| \alpha \right\|$$

This implies $\left(\mathrm{T}_{up}\right)_{\tilde{x}_w}$ has a bound and so $\left(\mathrm{T}_{up}\right)_{\tilde{x}_w}$ is continuous.

Taking,

$$\left\|\left(\left(\mathrm{T}_{up}\right)_{\tilde{x}_w}\right)_n-\left(\left(\mathrm{T}_{up}\right)_{\tilde{x}_w}\right)_m\right\|=\sup_{\|\tilde{\mathrm{X}}_{\mathrm{E}}\|\tilde{\le}1}\left\{\left\|\left[\left(\mathrm{T}_{up}\right)_n-\left(\mathrm{T}_{up}\right)_m\right]\left(\tilde{\mathrm{X}}_{\mathrm{E}}\right)\right\|\left(\tilde{x}_w\right)\right\}$$

Allowing $m\to\infty$ and n fixed,

$$\left\|\left(\left(\mathrm{T}_{up}\right)_{\tilde{x}_w}\right)_n-\left(\mathrm{T}_{up}\right)_{\tilde{x}_w}\right\|=\sup_{\|\tilde{\mathrm{X}}_{\mathrm{E}}\|\tilde{\le}1}\left\{\left\|\left[\left(\mathrm{T}_{up}\right)_n-\mathrm{T}_{up}\right]\left(\tilde{\mathrm{X}}_{\mathrm{E}}\right)\right\|\left(\tilde{x}_w\right)\right\}$$

$$=\sup_{\|\tilde{\mathrm{X}}_{\mathrm{E}}\|\tilde{\le}1}\left\|\left(\mathrm{T}_{up}\right)_n\left(\tilde{\mathrm{X}}_{\mathrm{E}}\right)-\mathrm{T}_{up}\left(\tilde{\mathrm{X}}_{\mathrm{E}}\right)\right\|\left(\tilde{x}_w\right)$$

$$=\left\|\left(\mathrm{T}_{up}\right)_n\left(\tilde{\mathrm{X}}_{\mathrm{E}}\right)-\mathrm{T}_{up}\left(\tilde{\mathrm{X}}_{\mathrm{E}}\right)\right\|$$

$$\tilde{\le}\,\grave{o}$$

Therefore $\mathrm{W}([0,1])$ is complete and hence $\mathrm{W}([0,1])$ is a Banach Space.

Lemma 4.5

Let $\mathrm{FS}(\tilde{\mathrm{Y}})$ be a fuzzy soft linear subspace of a fuzzy soft normed linear space $\mathrm{FS}(\tilde{\mathrm{X}})$, let $\tilde{\mathrm{X}}_{\mathrm{E}}\tilde{\in}\mathrm{FS}(\tilde{\mathrm{X}})$ be chosen in such a way that $\tilde{\mathrm{X}}_{\mathrm{E}}$ not in $\mathrm{FS}(\tilde{\mathrm{Y}})$. Define $\mathrm{FS}(\tilde{\mathrm{Y}}_0)=\mathrm{FS}(\tilde{\mathrm{Y}})+[\tilde{\mathrm{X}}_{\mathrm{E}}]$, the fuzzy soft linear subspace spanned by $\mathrm{FS}(\tilde{\mathrm{Y}})$ and $\tilde{\mathrm{X}}_{\mathrm{E}}$. Let T_{up} be a fuzzy soft linear operator on $\mathrm{FS}(\tilde{\mathrm{Y}})$ and its corresponding map $\left(\mathrm{T}_{up}\right)_{\tilde{y}_w}:[0,1]\to[0,1]$ be defined by $\left(\mathrm{T}_{up}\right)_{\tilde{y}_w}(\alpha)=\mathrm{T}_{up}\left(\tilde{\mathrm{Y}}_{\mathrm{E}}\right)\left(\tilde{y}_w\right)$ for all $\alpha\in[0,1]$ and $\tilde{\mathrm{Y}}_{\mathrm{E}}\left(\tilde{y}_w\right)=\alpha$ .Then $\left(\mathrm{T}_{up}\right)_{\tilde{y}_w}$ can be extended to $\left(\mathrm{T}_{up}\right)_{\tilde{y}_{w_0}}$ which maps from $[0,1]$ to $[0,1]$ such that $\left\|\left(\mathrm{T}_{up}\right)_{\tilde{y}_w}\right\|=\left\|\left(\mathrm{T}_{up}\right)_{\tilde{y}_{w_0}}\right\|$ where $\left(\mathrm{T}_{up}\right)_{\tilde{y}_{w_0}}(\alpha)=\left(\mathrm{T}_{up}\right)_0\left(\tilde{\mathrm{Y}}_0\right)_{\mathrm{E}}\left(\tilde{y}_{w_0}\right)$ and $\left(\mathrm{T}_{up}\right)_0$ is the extension of T_{up} .

Proof

Assume $\left\|\left(\mathrm{T}_{up}\right)_{\tilde{y}_w}\right\|=1$

Since $\mathrm{FS}(\tilde{\mathrm{Y}}_0)=\mathrm{FS}(\tilde{\mathrm{Y}})+[\tilde{\mathrm{X}}_{\mathrm{E}}]$, each $\left(\tilde{\mathrm{Y}}_0\right)_{\mathrm{E}}$ can be uniquely represented as

$$\left(\tilde{\mathrm{Y}}_0\right)_{\mathrm{E}}=\tilde{\mathrm{Y}}_{\mathrm{E}}+\tilde{k}.\tilde{\mathrm{X}}_{\mathrm{E}} \qquad (1)$$

This implies, $\left(\mathrm{T}_{up}\right)_0\left(\tilde{\mathrm{Y}}_0\right)_{\mathrm{E}}\left(\tilde{y}_{w_0}\right)=\left(\mathrm{T}_{up}\right)_{\tilde{y}_{w_0}}=\left(\mathrm{T}_{up}\right)_0\left[\tilde{\mathrm{Y}}_{\mathrm{E}}+\tilde{k}.\tilde{\mathrm{X}}_{\mathrm{E}}\right]$

$$\Rightarrow\quad \left(\mathrm{T}_{up}\right)_{\tilde{y}_{w_0}}(\alpha)=\left(\mathrm{T}_{up}\right)_0\left(\tilde{\mathrm{Y}}_{\mathrm{E}}\right)\left(\tilde{y}_w\right)+\left(\mathrm{T}_{up}\right)_0\left(\tilde{k}.\tilde{\mathrm{X}}_{\mathrm{E}}\right)\left(\tilde{x}_w\right)$$

$$\Rightarrow\quad \left(\mathrm{T}_{up}\right)_{\tilde{y}_{w_0}}(\alpha)=\left(\mathrm{T}_{up}\right)_0\left(\tilde{\mathrm{Y}}_{\mathrm{E}}\right)\left(\tilde{y}_w\right)+\tilde{k}\left(\left(\mathrm{T}_{up}\right)_0\right)_{\tilde{x}_w}(\alpha)$$

$$\left(\mathrm{T}_{up}\right)_{\tilde{y}_{w_0}}(\alpha)=\left(\mathrm{T}_{up}\right)_{\tilde{y}_w}(\alpha)+\tilde{k}.\grave{o},\text{ where }\grave{o}\in[0,1] \qquad (2)$$

When $\left(\mathrm{T}_{up}\right)_{\tilde{y}_{w_0}}$ is restricted so that $\left(\mathrm{T}_{up}\right)_0$ is restricted on $\mathrm{FS}(\tilde{\mathrm{Y}})$ it equals T_{up} on $\mathrm{FS}(\tilde{\mathrm{Y}})$.

Hence $\left(\mathrm{T}_{up}\right)_{\tilde{y}_{w_0}}$ is the extension of $\left(\mathrm{T}_{up}\right)_{\tilde{y}_w}$.

As $\left(\mathrm{T}_{up}\right)_{\tilde{y}_{w_0}}(\alpha)=\left(\mathrm{T}_{up}\right)_{\tilde{y}_w}(\alpha)+\tilde{k}.\grave{o}$

$$\left\|\left(\mathrm{T}_{up}\right)_{\tilde{y}_{w_0}}\right\|=\left\|\left(\mathrm{T}_{up}\right)_0\right\|\left(\tilde{y}_{w_0}\right)$$

$$\left\|\left(\mathrm{T}_{up}\right)_{\tilde{y}_{w_0}}\right\| = \sup\left\{\frac{\left\|\left(\mathrm{T}_{up}\right)_0\left(\tilde{\mathrm{Y}}_0\right)_{\mathrm{E}}\right\|}{\left\|\left(\tilde{\mathrm{Y}}_0\right)_{\mathrm{E}}\right\|}\left(\tilde{y}_{w_0}\right) : \left\|\left(\tilde{\mathrm{Y}}_0\right)_{\mathrm{E}}\right\| \neq \tilde{0}\right\}$$

To prove that $\left\|\left(\mathrm{T}_{up}\right)_{\tilde{y}_{w_0}}\right\| = 1$

$$\text{i.e., } \sup\left\{\frac{\left\|\left(\mathrm{T}_{up}\right)_0\left(\tilde{\mathrm{Y}}_0\right)_{\mathrm{E}}\right\|}{\left\|\left(\tilde{\mathrm{Y}}_0\right)_{\mathrm{E}}\right\|}\left(\tilde{y}_{w_0}\right) : \left\|\left(\tilde{\mathrm{Y}}_0\right)_{\mathrm{E}}\right\| \neq \tilde{0}\right\} = 1$$

$$\text{i.e., } \frac{\left\|\left(\mathrm{T}_{up}\right)_0\left(\tilde{\mathrm{Y}}_0\right)_{\mathrm{E}}\right\|}{\left\|\left(\tilde{\mathrm{Y}}_0\right)_{\mathrm{E}}\right\|}\left(\tilde{y}_{w_0}\right) \tilde{\leq} 1$$

$$\left\|\left(\mathrm{T}_{up}\right)_0\left(\tilde{\mathrm{Y}}_0\right)_{\mathrm{E}}\right\|\left(\tilde{y}_{w_0}\right) \tilde{\leq} \left\|\left(\tilde{\mathrm{Y}}_0\right)_{\mathrm{E}}\right\|$$

This implies, $\left\|\left(\mathrm{T}_{up}\right)_{\tilde{y}_{w_0}}(\alpha)\right\| \tilde{\leq} \left\|\left(\tilde{\mathrm{Y}}_0\right)_{\mathrm{E}}\right\|$

Using (1) and (2)

$$\left\|\left(\mathrm{T}_{up}\right)_{\tilde{y}_w}(\alpha) + \tilde{k}.\grave{o}\right\| \tilde{\leq} \left\|\tilde{\mathrm{Y}}_{\mathrm{E}} + \tilde{k}.\tilde{\mathrm{X}}_{\mathrm{E}}\right\|$$

$$-\left\|\tilde{\mathrm{Y}}_{\mathrm{E}} + \tilde{k}.\tilde{\mathrm{X}}_{\mathrm{E}}\right\| \;\tilde{\leq}\; \left(\mathrm{T}_{up}\right)_{\tilde{y}_w}(\alpha) + \tilde{k}.\grave{o} \;\tilde{\leq}\; \left\|\tilde{\mathrm{Y}}_{\mathrm{E}} + \tilde{k}.\tilde{\mathrm{X}}_{\mathrm{E}}\right\|$$

$$-\left(\mathrm{T}_{up}\right)_{\tilde{y}_w}(\alpha) - \left\|\tilde{\mathrm{Y}}_{\mathrm{E}} + \tilde{k}.\tilde{\mathrm{X}}_{\mathrm{E}}\right\| \;\tilde{\leq}\; \tilde{k}.\grave{o} \;\tilde{\leq}\; -\left(\mathrm{T}_{up}\right)_{\tilde{y}_w}(\alpha) + \left\|\tilde{\mathrm{Y}}_{\mathrm{E}} + \tilde{k}.\tilde{\mathrm{X}}_{\mathrm{E}}\right\|$$

$$-\left(\mathrm{T}_{up}\right)_{\tilde{y}_w}\left(\frac{1}{\tilde{k}}.\alpha\right) - \frac{1}{\tilde{k}}\left\|\tilde{\mathrm{Y}}_{\mathrm{E}} + \tilde{k}.\tilde{\mathrm{X}}_{\mathrm{E}}\right\| \;\tilde{\leq}\; \grave{o} \;\tilde{\leq}\; -\left(\mathrm{T}_{up}\right)_{\tilde{y}_w}\left(\frac{1}{\tilde{k}}.\alpha\right) + \frac{1}{\tilde{k}}\left\|\tilde{\mathrm{Y}}_{\mathrm{E}} + \tilde{k}.\tilde{\mathrm{X}}_{\mathrm{E}}\right\| \qquad (3)$$

Let $\tilde{\mathrm{Y}}_{\mathrm{E}}, \tilde{\mathrm{Y}}'_{\mathrm{E}} \tilde{\in} \mathrm{F\,S}\left(\tilde{\mathrm{Y}}\right)$

$$\begin{aligned}
\mathrm{T}_{up}\left(\tilde{\mathrm{Y}}_{\mathrm{E}}\right)\left(\tilde{y}_w\right) - \mathrm{T}_{up}\left(\tilde{\mathrm{Y}}'_{\mathrm{E}}\right)\left(\tilde{y}_w\right) &= \mathrm{T}_{up}\left[\left(\tilde{\mathrm{Y}}_{\mathrm{E}}\right) - \left(\tilde{\mathrm{Y}}'_{\mathrm{E}}\right)\right]\left(\tilde{y}_w\right) \\
&\tilde{\leq} \left\|\mathrm{T}_{up}\left(\tilde{\mathrm{Y}}_{\mathrm{E}} - \tilde{\mathrm{Y}}'_{\mathrm{E}}\right)\left(\tilde{y}_w\right)\right\| \\
&\tilde{\leq} \left\|\mathrm{T}_{up}\right\|\left\|\left(\tilde{\mathrm{Y}}_{\mathrm{E}} - \tilde{\mathrm{Y}}'_{\mathrm{E}}\right)\left(\tilde{y}_w\right)\right\| \\
&= \left\|\tilde{\mathrm{Y}}_{\mathrm{E}}\left(\tilde{y}_w\right) - \tilde{\mathrm{Y}}'_{\mathrm{E}}\left(\tilde{y}_w\right)\right\| \\
&= \left\|\tilde{\mathrm{Y}}_{\mathrm{E}}\left(\tilde{y}_w\right) - \tilde{\mathrm{X}}_{\mathrm{E}}\left(\tilde{x}_w\right) + \tilde{\mathrm{X}}_{\mathrm{E}}\left(\tilde{x}_w\right) - \tilde{\mathrm{Y}}'_{\mathrm{E}}\left(\tilde{y}_w\right)\right\| \\
&\tilde{\leq} \left\|\tilde{\mathrm{Y}}_{\mathrm{E}}\left(\tilde{y}_w\right) - \tilde{\mathrm{X}}_{\mathrm{E}}\left(\tilde{x}_w\right)\right\| + \left\|\tilde{\mathrm{Y}}'_{\mathrm{E}}\left(\tilde{y}_w\right) - \tilde{\mathrm{X}}_{\mathrm{E}}\left(\tilde{x}_w\right)\right\| \\
\mathrm{T}_{up}(\alpha) - \mathrm{T}_{up}(\alpha) &\tilde{\leq} \left\|\tilde{\mathrm{Y}}_{\mathrm{E}}\left(\tilde{y}_w\right) - \tilde{\mathrm{X}}_{\mathrm{E}}\left(\tilde{x}_w\right)\right\| + \left\|\tilde{\mathrm{Y}}'_{\mathrm{E}}\left(\tilde{y}_w\right) - \tilde{\mathrm{X}}_{\mathrm{E}}\left(\tilde{x}_w\right)\right\|
\end{aligned}$$

$$-\mathrm{T}_{up}(\alpha) - \left\|\tilde{\mathrm{Y}}_{\mathrm{E}}\left(\tilde{y}_w\right) - \tilde{\mathrm{X}}_{\mathrm{E}}\left(\tilde{x}_w\right)\right\| \;\tilde{\leq}\; -\mathrm{T}_{up}(\alpha) + \left\|\tilde{\mathrm{Y}}'_{\mathrm{E}}\left(\tilde{y}_w\right) - \tilde{\mathrm{X}}_{\mathrm{E}}\left(\tilde{x}_w\right)\right\| \qquad (4)$$

This is true for all $\tilde{\mathrm{Y}}_{\mathrm{E}} \tilde{\in} \mathrm{F\,S}\left(\tilde{\mathrm{Y}}\right)$.

Therefore if $\tilde{r}_{e_1}, \tilde{r}_{e_2}$ are two fuzzy real numbers then (4) becomes $\tilde{r}_{e_1} \tilde{\leq} \tilde{r}_{e_2}$.

Choose $\tilde{r}_e$, a fuzzy soft real number such that $\tilde{r}_{e_1} \tilde{\leq} \tilde{r}_e \tilde{\leq} \tilde{r}_{e_2}$.

The required inequality (3) is satisfied.

Thus $\left\|\left(\mathrm{T}_{up}\right)_{\tilde{y}_w}\right\| = \left\|\left(\mathrm{T}_{up}\right)_{\tilde{y}_{w_0}}\right\|$.

4.6 (The Hahn- Banach Theorem)

Let $\mathrm{FS}(\tilde{\mathrm{Y}})$ be a linear subspace of a normed linear space $\mathrm{FS}(\tilde{\mathrm{X}})$ and let T_{up} be a fuzzy soft linear operator defined on $\mathrm{FS}(\tilde{\mathrm{Y}})$. Then its corresponding map $(\mathrm{T}_{up})_{\tilde{y}_w}:[0,1]\to[0,1]$ defined by $(\mathrm{T}_{up})_{\tilde{y}_w}(\alpha)=\mathrm{T}_{up}(\tilde{\mathrm{Y}}_{\mathrm{E}})(\tilde{y}_w)$ for all $\alpha\in[0,1]$ and $\tilde{\mathrm{Y}}_{\mathrm{E}}(\tilde{y}_w)=\alpha$, can be extended to an operator $(\mathrm{T}_{up})_{\tilde{y}_{w_0}}$ defined on the whole interval $[0,1]$ such that $\left\|(\mathrm{T}_{up})_{\tilde{y}_w}\right\|=\left\|(\mathrm{T}_{up})_{\tilde{y}_{w_0}}\right\|$, where $(\mathrm{T}_{up})_{\tilde{y}_{w_0}}(\alpha)=(\mathrm{T}_{up})_0(\tilde{\mathrm{Y}}_0)_{\mathrm{E}}(\tilde{y}_{w_0})$ and $(\mathrm{T}_{up})_0$ is the extension of T_{up} which is defined on the whole space $\mathrm{FS}(\tilde{\mathrm{X}})$.

Proof

The set of all extensions of $(\mathrm{T}_{up})_{\tilde{y}_w}$ to the fuzzy soft linear operator $(\mathrm{S}_{up})_{\tilde{y}_{w_i}}$ with the same norm defined on $[0,1]$ is clearly a partially ordered set with respect to the relation $(\mathrm{S}_{up})_{\tilde{y}_{w_1}}\tilde{\leq}(\mathrm{S}_{up})_{\tilde{y}_{w_2}}$ means that the domain of $(\mathrm{S}_{up})_{\tilde{y}_{w_1}}$ is contained in the domain of $(\mathrm{S}_{up})_{\tilde{y}_{w_2}}$.

This implies $(\mathrm{S}_{up})_1(\tilde{\mathrm{X}}_{\mathrm{E}})_1(\tilde{x}_{w_1})=(\mathrm{S}_{up})_2(\tilde{\mathrm{X}}_{\mathrm{E}})_1(\tilde{x}_{w_1})$

where $((\mathrm{S}_{up})_1)_{\tilde{x}_{w_1}}(\alpha)=((\mathrm{S}_{up})_2)_{\tilde{x}_{w_1}}(\alpha),\forall(\tilde{\mathrm{X}}_{\mathrm{E}})_1(\tilde{x}_{w_1})=\alpha\in[0,1]$.

The union of any chain of extensions is also an extension and is therefore an upper bound for the chain.

Zorn's lemma states that, if P is a partially ordered set in which every chain has an upper bound, then P possesses a maximal element.

This implies that there exists a maximal extension $(\mathrm{T}_{up})_{\tilde{y}_{w_0}}$.

The proof completes by assuming the domain of $(\mathrm{T}_{up})_{\tilde{y}_{w_0}}$ must be the entire space $[0,1]$, for otherwise, if $\tilde{\mathrm{X}}_{\mathrm{E}}$ is an element not in $\mathrm{FS}(\tilde{\mathrm{Y}})$ then $(\mathrm{T}_{up})_{\tilde{y}_{w_0}}$ can be extended to an operator defined on $\mathrm{FS}(\tilde{\mathrm{X}})$, which is a contradiction to that $(\mathrm{T}_{up})_{\tilde{y}_{w_0}}$ is the maximal extension.

Therefore $(\mathrm{T}_{up})_{\tilde{y}_{w_0}}$ is the extension defined on the whole interval $[0,1]$ with the property $\left\|(\mathrm{T}_{up})_{\tilde{y}_w}\right\|=\left\|(\mathrm{T}_{up})_{\tilde{y}_{w_0}}\right\|$.

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