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APPROXIMATE SOLUTION OF FREDHOLM INTEGRAL EQUATION OF THE SECOND KIND USING A COMBINE METHOD

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ARTICLE INFO | **ABSTRACT**

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A combine method is given for the approximate solution of Fredholm integral equations of the second kind. The proposed method based on the iteration techniques, while the kernel and the given known functions of the integral equation are approximated by Maclaurin polynomials of degree n with error estimations. The convergence of the solution is studied and the conditions for the convergent solution is given. An algorithm of the obtained solution is established via matrices. Thus reducing the required solution so that only one coefficient matrix is required to be computed. Therefore, computational complexity can be considerably reduced and much computational time can be saved. The new proposed approach needs a small number of iterations to provide an exact result, that proofs the power of the presented combine method.

. **KEYWORDS**: Approximate Solution Of Fredholm Integral Equation

1. Introduction

Many methods of solving Fredholm integral equation of the second kind have been developed in recent years [1,3,6,7,13].The objective of this paper is to establish a new method that can be easily programmed on Matlab.

Computational modifications are performed on the iterative algorithm presented in [9,10], where the procedure beginning by replacing the kernel of an integral equation approximately by a degenerate kernel [4,5,8] in a matrix form using Maclaurin polynomial of degree n, whereas the given function is approximated by Maclaurin polynomial of the same degree n with maximum error estimation[2]. Owing to the simplicity of some operational matrix presentation, the solution is reduced to the computation of only one matrix.

Despite of the advantages of methods [4,5] there was apparent higher cost comparing with the present method, that minimizes the computational effort and smooth the round-off errors out. Due to the simple form of the obtained solution, the present method may be generalized to solve both second kind and well-posed singular integral equations of the first kind [11,12].

2. The proposed Combine method

Consider the Fredholm integral equation of the second kind

$$\phi(x)=\int_{\alpha}^{\beta} K(x,y)\phi(y)dy+f(x) \quad \forall q\geq 1 \quad \forall q\geq 1 \qquad (1)$$

where the function $f(x)$ and the kernel $K(x,y)$ are given and differentiable. The kernel $K(x,y)$ is defined in the square $\Omega=\{\alpha\leq x\leq\beta,\alpha\leq y\leq\beta\}$ in the $xy-$ plane. The function $\varphi(x)$ is the unknown required solution.

The given Algorithm starts with an initial approximation $\varphi^{(0)}(x)$ to the solution $\varphi(x)$ of integral equation (1) and then generates a sequence of solutions

$\{\varphi^{(q)}(x)\}_{q=0}^{\infty}$ that converges to $\varphi(x)$ such that $\left\|\dfrac{\phi^{(q)}(x)-\phi^{(q-1)}(x)}{\phi^{(q)}(x)}\right\|<\delta;\delta>0$

After the initial solution $\varphi^{(0)}(x)$ is chosen, the sequences of approximate solutions can be generated by computing

$$\phi^{(q)}(x)=f(x)+\int_{\alpha}^{\beta} K(x,y)\phi^{(q-1)}(y)dy \quad \forall q\geq 1 \qquad (2)$$

If we begin by $\varphi^{(0)}(x)=0$, then we get

$$\varphi^{(q)}(x)=f(x)+\sum_{s=1}^{q}\int_{\alpha}^{\beta} K^{(s)}(x,y)f(y)dy \quad \forall q\geq 1 \qquad (3)$$

where the iterated kernels $K^{(s)}(x,y)$ can be found by the recurrence form

$$K^{(s)}(x,y)=\int_{\alpha}^{\beta} K(x,z)K^{(s-1)}(z,y)dz \;\; ; \;\; s\geq 2 \quad \forall q\geq 1 \qquad (4)$$

where

$$K^{(1)}(x,y)=K(x,y)$$

Now, Approximating the kernel $K(x,y)$ by using Maclaurin polynomial of degree n , we get the matrix form

$$K(x,y)=\mathbf{P}^{t}(y)\mathbf{L}(\mathbf{n})\mathbf{K}\mathbf{L}(\mathbf{n})\mathbf{P}(x) \qquad (5)$$

where $\mathbf{K}=\left(k_{ij}\right)$ 1s an $(n+1)\times(n+1)$ coefficients matrix whose entries are defined by

$$k_{ij}=\begin{cases}\left.\dfrac{\partial^i\partial^j K(x,y)}{\partial x^i\partial y^j}\right|_{(x,y)=(0,0)} & ;i+j\le n \\ 0 & ;i+j>n\end{cases}\qquad \vee\, i,j=\overline{o,n} \tag{6}$$

and the matrix $\mathbf{L(n)}$ is an $(n+1)\times(n+1)$ matrix defined by

$$\mathbf{L(n)}=\mathbf{diag}\begin{pmatrix}\dfrac{1}{0!} & \dfrac{1}{1!} & \dots & \dfrac{1}{n!}\end{pmatrix} \tag{7}$$

The matrix $\mathbf{P(x)}$ of order $(n+1)\times(1)$ is defined to be

$$\mathbf{P}^{t}(\mathbf{x})=\left[\mathbf{p}_{o}(x)\quad \mathbf{p}_{1}(x)\quad \dots\quad \mathbf{p}_{n}(x)\right] \tag{8}$$

Where $\qquad \mathbf{p_i}(\mathbf{x})=x^{i}\quad \textbf{for}\ i=\overline{0;n}$

Now, we define the Maclaurin operational integration matrix, H, to be

$$\mathbf{H}=\int_{\alpha}^{\beta}\mathbf{P}(y)\mathbf{P}^{t}(y)dy \tag{9}$$

$$=\beta\ \mathbf{B}(\beta)\mathbf{Hilb}\ \mathbf{B}(\beta)-\alpha\ \mathbf{B}(\alpha)\mathbf{Hilb}\ \mathbf{B}(\alpha)$$

Where

$$\mathbf{B}(\alpha)=\mathbf{diag}\begin{pmatrix}1 & \alpha & \alpha^2 & \dots & \alpha^n\end{pmatrix};\ \mathbf{B}(\beta)=\mathbf{diag}\begin{pmatrix}1 & \beta & \beta^2 & \dots & \beta^n\end{pmatrix} \tag{10}$$

Here **Hilb** is the well - known Hilbert matrix of order $(n+1)\times(n+1)$ with elements $\mathbf{Hilb}_{ij}=(i+j-1)^{-1}$. Substituting (5) into (4) and by virtue of (9) the iterated kernels $K^{(s)}(x,y)$ become

$$K^{(s)}(x,y)=\mathbf{P}^{t}(x)\Big[\big(\mathbf{L}(\mathbf{n})\ \mathbf{K}\,\mathbf{L}(\mathbf{n})\big)^{t}\mathbf{H}\Big]^{(s-1)}\big(\mathbf{L}(\mathbf{n})\mathbf{K}\,\mathbf{L}(\mathbf{n})\big)^{t}\mathbf{P}(y) \tag{11}$$

Then, approximating the given function $f(x)$ in Maclourin polynomial of degree n yields

$$\begin{aligned} &f(x)=\mathbf{P}^{t}(x)\ \mathbf{F}\ +\Re(x);\ \mathbf{F}^{t}=\begin{bmatrix} f_o & f_1 & \ldots & f_n\end{bmatrix};\\ &\Re(x)=r_n(x)I \end{aligned} \tag{12}$$

Where $$f_i=\frac{1}{i!}\left\{\frac{d^{i}f(x)}{dx^{i}}\right\}_{x=0}\quad ;\ i=\overline{0,n}$$

Here the matrix $\Re(x)$ denotes the matrix of the maximum error estimated at whole interval $[\alpha,\beta]$, where $r_n(x)$

$$r_{n}(x)\le\frac{\lambda_{n+1}}{(n+1)!}\gamma^{n+1};\quad \lambda_{n+1}=\max_{\xi\in[\alpha,\beta]}\left|f^{(n+1)}(\xi)\right|<\infty,$$

and $\gamma=\max\{\alpha-x,x-\beta\}$, and I the unit matrix of order 1.

Now, Substituting (11) and (12) into (3) we find that

$$\begin{aligned} &\phi^{(q)}(x)=\mathbf{P}^{t}(x)\mathbf{F}+\Re(x)\\ &+\sum_{s=1}^{q}\int_{\alpha}^{\beta}\mathbf{P}^{t}(x)\left[\tilde{\mathbf{K}}^{t}\mathbf{H}\right]^{s-1}\tilde{\mathbf{K}}^{t}\ \mathbf{P}(y)\left[\mathbf{P}^{t}(y)\mathbf{F}+\Re(y)\right]dy \end{aligned} \tag{13}$$

$$\phi^{(q)}(x)=\mathbf{P}^{\mathrm{t}}(x)\mathbf{F}+\Re(x) +\sum_{s=1}^{q}\int_{\alpha}^{\beta}\mathbf{P}^{\mathbf{t}}(x)\left[\tilde{\mathbf{K}}^{\mathrm{t}}\mathbf{H}\right]^{\mathbf{s-1}}\tilde{\mathbf{K}}^{\mathbf{t}}\,\mathbf{P}(y)\mathbf{P}^{\mathbf{t}}(y)\mathbf{F}dy +\sum_{s=1}^{q}\int_{\alpha}^{\beta}\mathbf{P}^{\mathbf{t}}(x)\left[\tilde{\mathbf{K}}^{\mathrm{t}}\mathbf{H}\right]^{\mathbf{s-1}}\tilde{\mathbf{K}}^{\mathbf{t}}\,\mathbf{P}(y)\,\Re(y)dy\;\forall\, q\geq 1 \tag{14}$$

$$\phi^{(q)}(x)=\mathbf{P}^{\mathrm{t}}(x)\mathbf{F}+\Re(x)+\sum_{s=1}^{q} P^{t}(x)[\tilde{\mathbf{K}}^{\mathrm{t}}\mathbf{H}]^{(\mathrm{s})}\mathbf{F} +\sum_{s=1}^{q}\mathbf{P}^{\mathbf{t}}(x)\left[\tilde{\mathbf{K}}^{\mathrm{t}}\mathbf{H}\right]^{(\mathbf{s-1})}\tilde{\mathbf{K}}^{\mathbf{t}}\left[\int_{\alpha}^{\beta}\mathbf{P(y)}\,dy\right]\Re(y) \tag{15}$$

$$\text{Let } \tilde{\mathbf{P}}=\int_{\alpha}^{\beta}\mathbf{P}(y)dy=\begin{bmatrix}(\beta-\alpha)\\ (\beta-\alpha)^2/2\\ \vdots\\ (\beta-\alpha)^{n+1}/n+1\end{bmatrix},\quad n\geq 0 \tag{16}$$

$$\phi^{(q)}(x)=\mathbf{P}^{\mathrm{t}}(x)\mathbf{F}+\Re(x)+\sum_{s=1}^{q} P^{t}(x)[\tilde{\mathbf{K}}^{\mathrm{t}}\mathbf{H}]^{(\mathrm{s}-1)}\tilde{\mathbf{K}}^{\mathrm{t}}\left[\mathbf{HF}+\tilde{\mathbf{P}}\Re(y)\right] \tag{17}$$

$$\phi^{(q)}(x)=\mathbf{P}^{\mathrm{t}}(x)\left[\mathbf{F}+\sum_{s=1}^{q}[\tilde{\mathbf{K}}^{\mathrm{t}}\mathbf{H}]^{(\mathrm{s}-1)}\tilde{\mathbf{K}}^{\mathrm{t}}\left[\mathbf{HF}+\tilde{\mathbf{P}}\Re(y)\right]\right]+\Re(x) \tag{18}$$

Let $\psi^{(q)}$ is the $(\mathbf{n+1})\times(\mathbf{n+1})$ iterative coefficients vector defined by

$$\psi^{(q)}=\sum_{s=1}^{q}[\tilde{\mathbf{K}}^{\mathrm{t}}\mathbf{H}]^{(\mathrm{s}-1)}\tilde{\mathbf{K}}^{\mathrm{t}}$$

Then we have

$$\phi^{(q)}(x)=\mathbf{P}^{\mathrm{t}}(x)\left[\mathbf{F}+\psi^{(q)}\left[\mathbf{HF}+\tilde{\mathbf{P}}\Re(y)\right]\right]+\Re(x) \tag{19}$$

If $C = K^{-1}H$,then the approximate solution $\phi^{(q)}(x)$ converges to the exact $\phi(x)$ if one of the following three conditions is satisfied

$$\|C\|_{\infty} = \max_{1 \le i \le n} \sum_{j=1}^{n} |C_{ij}| < 1;$$

$$\|C\|_{\infty} = \max_{1 \le j \le n} \sum_{i=1}^{n} |C_{ij}| < 1;$$

$$\|C\|_{2} = \left(\rho\left(CC^{t}\right)\right)^{2} < 1$$

Where $\rho\left(CC^{t}\right)$ is the spectral radius of CC^{t} .

That is, to find the iterative solutions $\varphi^{(q)}(x)$ of integral equation (1) it is required only to compute the matrix K given by (6)

3. Computational results

The computational results are made on a personal computer using matlab 2010.

3.1 Example(1) :

Solve the integral equation $\phi(x) = 3 - \frac{4}{3}x + \int_{0}^{1} xt\,\phi(t)\,dt$

whose exact solution is given by $\phi(x) = 3 + \frac{1}{4}x$

Table (1) : The numerical solutions without error estimation term

N	$\phi(x)$	N	$\phi(x)$
1	**0.00833 *x^5 - 0.16666 *x^3 + 1.0 *x**	5	**0.000001 *x^9 + 0.00003 *x^7 + 0.00848 *x^5 - 0.16504 *x^3 + 0.99272 *x**
2	**0.000001 *x^9 + 0.00002 *x^7 + 0.00855 *x^5 - 0.16682 *x^3 + 1.00316 *x**	6	**0.000001 *x^9 + 0.00003 *x^7 + 0.00848 *x^5 - 0.16505 *x^3 + 0.99275 *x**
3	**0.000001 *x^9 + 0.00003 *x^7 + 0.00851 *x^5 - 0.16549 *x^3 + 0.99546 *x**	7	**0.000001 *x^9 + 0.00003 *x^7 + 0.00848 *x^5 - 0.16505 *x^3 + 0.99277 *x**
4	**0.000001 *x^9 + 0.00003 *x^7 + 0.00849 *x^5 - 0.16509**		

	***x^3 + 0.99305 *x**		

The CPU time is equal to 4.0092 Seconds

As the above table show, we reached the exact solution after 7 iterations.

Figure (1) : The exact and numerical solutions without error estimation term

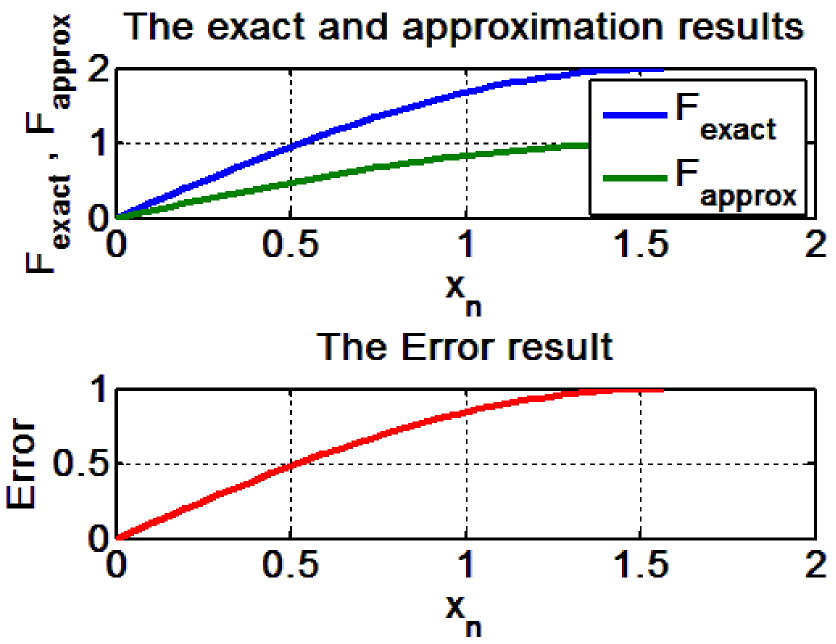

Figure (1)

Table (2) : The exact and numerical solutions with error estimation term

N	$\phi(x)$	N	$\phi(x)$
1	**0.000001 *x^9 + 0.00013 *x^7 + 0.01489 *x^5 - 0.20998 *x^3 + 0.98798 *x**	4	**0.000001 *x^9 + 0.00013 *x^7 + 0.01489 *x^5 +0.20998 *x^3 + 1.37202 *x**
2	**0.000001 *x^9 + 0.00013 *x^7 + 0.01489 *x^5 + 0.20998 *x^3 + 1.34386 *x**	5	**0.000001 *x^9 + 0.00013 *x^7 + 0.01489 *x^5 + 0.20998 *x^3 + 1.37272 *x**
3	**0.000001 *x^9 + 0.00013 *x^7 + 0.01489 *x^5 + 0.20998 *x^3 + 1.36889 *x**	6	**0.000001 *x^9 + 0.00013 *x^7 + 0.01489 *x^5 + 0.20998 *x^3 + 1.37289 *x**

The CPU time is equal to 3. 9574 Seconds

As the above table show, we reached the exact solution after 6 iterations

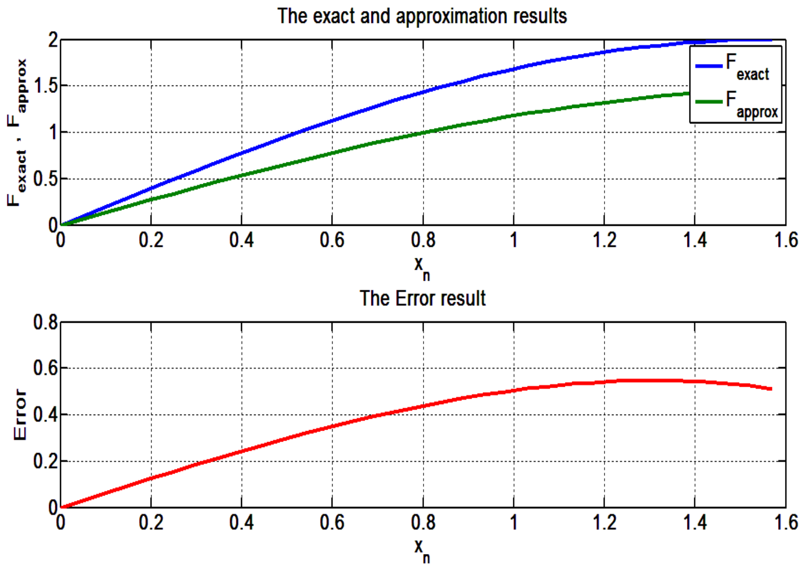

Figure (2)

3.2 Example(2) : Solve the integral equation $\phi(x) = e^x - 1 + \int_0^1 t\,\phi(t)\,dt$

Whose exact solution is given by $\phi(x) = e^x$

Table (3) : The numerical solutions without error estimation term

N	$\phi(x)$	N	$\phi(x)$
1	**0.00833*x^2 + 0.04166*x + 0.20833**	9	**0.00833*x^2 + 0.04166*x+ 0.30725**
2	**0.00833*x^2 + 0.04166*x + 0.25798**	10	**0.00833*x^2 + 0.04166*x + 0.30744**
3	**0.00833*x^2 + 0.04166*x+ 0.28281**	11	**0.00833*x^2 + 0.04166*x+ 0.30754**
4	**0.00833*x^2 + 0.04166*x+ 0.28281**	12	**0.00833*x^2 + 0.04166*x+ 0.30759**
5	**0.00833*x^2 + 0.04166*x+ 0.29522**	13	**0.00833*x^2 + 0.04166*x + 0.30761**
6	**0.00833*x^2 + 0.04166*x + 0.30143**	14	**0.00833*x^2 + 0.04166*x+ 0.30762**
7	**0.00833*x^2 + 0.04166*x + 0.30453**	15	**0.00833*x^2 + 0.04166*x+ 0.30763**
8	**0.00833*x^2 + 0.04166*x+ 0.30608**		

The CPU time is equal to 1.3416 Seconds

As the above table show, we reached the exact solution after 15 iterations.

Figure (3) : The exact and numerical solutions without error estimation term

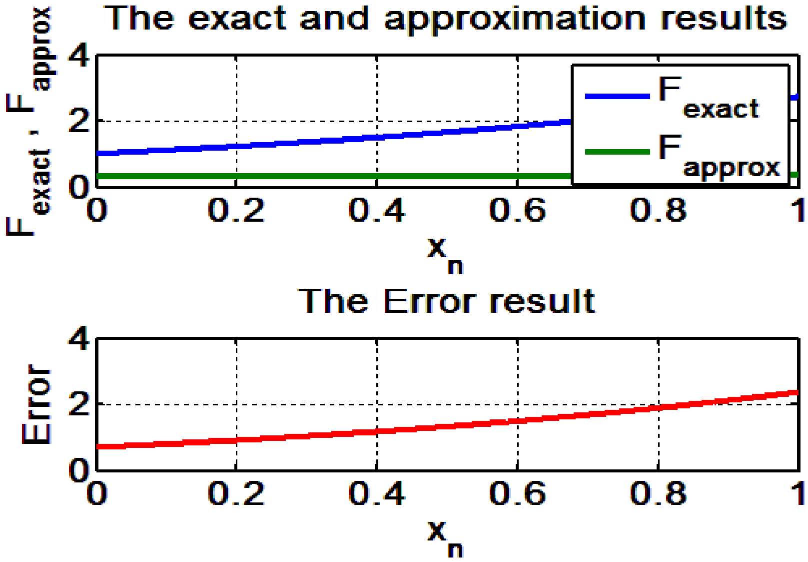

Figure (3)

Table (4) : The numerical solutions with error estimation term

N	$\phi(x)$	N	$\phi(x)$
1	**0.40036*x^2 + 0.61666*x+ 0.21457**	7	**0.40036*x^2 + 0.61666*x+ 0.35687**
2	**0.40036*x^2 + 0.61666*x+ 0.23684**	8	**0.40036*x^2 + 0.61666*x+ 0.37018**
3	**0.40036*x^2 + 0.61666*x+ 0.26923**	9	**0.40036*x^2 + 0.61666*x+ 0.39034**
4	**0.40036*x^2 + 0.61666*x+ 0.29008**	10	**0.40036*x^2 + 0.61666*x+ 0.40032**
5	**0.40036*x^2 + 0.61666*x+ 0.31608**	11	**0.40036*x^2 + 0.61666*x+ 0.40568**
6	**0.40036*x^2 + 0.61666*x+ 0.33002**	12	**0.40036*x^2 + 0.61666*x+ 0.40762**

The CPU time is equal to 1.2606 Seconds

As the above table show, we reached the exact solution after 12 iterations.

Figure (4) : The exact and numerical solutions with error estimation term

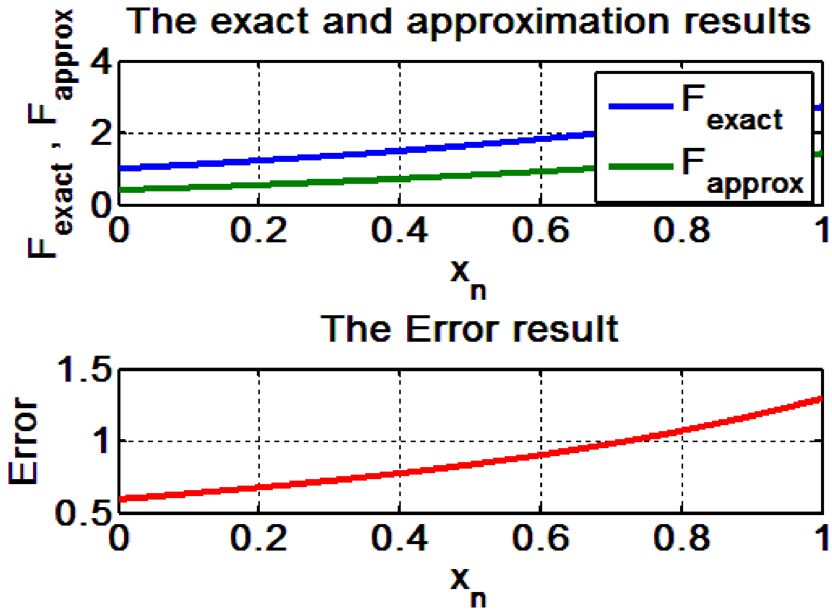

Figure (4)

5. Conclusion

A simple method for the solution of Fredholm Integral Equations of the second kind has been presented. The given method gives a very simple form for the iterated kernels via the well - known Hilbert matrix. Thus, the approximate solution of an integral equation of the second kind can be reduced to the solution of a matrix equation, whereas only one coefficient matrix is required to be computed. Therefore, computational complexity can be considerably reduced and much computational time can be saved. The new proposed approach needs a small number of iterations to provide an exact result that proofs the power of the presented method, and stimulates to find out the relation between the integral equations and Hilbert Matrix. Briefly words the Advantages of the given method maybe be concluded as

- Minimize the CPU time in a spectacular way.
- Use less number of iterations to reach the exact solution
- Easy to compute because only one matrix is required to find the approximate solution
- It is more powerful if the kernel function is complicated as in this method we differentiate the kernel to get the solution but in the traditional iterated kernel method we integrate

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