On π^g-closed sets in topological spaces

V. Jeyanthi*, C. Janaki, F. Soumya*****

* Assistant professor, Department of Mathematics, Sree Narayana Guru College, Coimbatore-105
jeyanthi_sngc@yahoo.com
** Department of Mathematics, L.R.G. Govt. Arts College for Women, Tirupur-4
janakicsekar @yahoo.com
*** Department of Mathematics, Sree Narayana Guru College, Coimbatore-105
Soumyaf06@gmail.com

ARTICLE INFO | ABSTRACT

Corresponding Author:
V. Jeyanthi

The aim of this paper is to introduce a new class of closed sets in topological spaces called π^g-closed sets and obtain some of its characteristics. Also, the concept of continuity called π^g-continuity is defined and obtained some of its properties.

π^g-closed sets, π^g-$T_{1/2}$-space and π^g-continuous function.

1. Introduction

Topology is the one of the important area in mathematics. It stands one among the most recent in whole of mathematics. The notion of closed set is fundamental in the study of topological spaces. N. Levine[12] introduced the generalized closed sets, a super class of closed sets in 1970. After the advent of g-closed sets, many variations of g-closed sets were introduced and investigated by many topologists. In 1968, Zaitsev[19] introduced the notion of π-open sets as a finite union of regular open sets. J. Dontchev T. Noiri[5] introduced and studied πg-closed set in topological spaces. Jeyanthi V and Janaki C[9] introduced its stronger form called πgr-closed set in topological spaces. The weaker of πg-closed set was introduced and studied by [16],[2],[8],[18]. Here, we are introducing a new class of closed set in topological spaces called π^g- closed set which is independent of πg-closed set and obtain some of its characteristics.

2. Preliminaries

Throughout this paper (X, τ), (Y, σ) and (Z, η) represent topological spaces on which no separation axioms are assumed unless otherwise mentioned. For any subset A of a space (X, τ),cl(A) and int(A) denote the closure of A and interior of A respectively. The topological space (X, τ) will be replaced by X if there is no chance of confusion.

Let us recall the following definitions which we shall require in sequel.

Definition 2.1: A subset A of a space X is called

1) a pre-open set [1] if A⊂int(cl(A)) and a pre-closed set if cl(int(A)) ⊂ A,

2) a semi-open set [11]if $A\subset cl(int(A))$ and a semi-closed set if $int(cl(A))\subset A$,

3) a α-open set [15] if $A\subset int(cl(int(A)))$and a α-closed set if $cl(int(cl(A)))\subset A$,

4) a semi-pre-open[1] if $A\subset cl(int(cl(A)))$ and a semi-pre-closed set if $int(cl(int(A)))\subset A$,

5) a regular open set if $A=int(cl(A))$ and a regular closed[13] set if$A=cl(int(A))$.

The intersection of all semi-closed(resp. α-closed, pre-closed, semipre-closed, regular closed and b-closed) subsets of X containing A is called the semi closure(resp. α-closure, pre-closure, semi pre-closure, regular closure and b-closure) of A and is denoted by sclA(resp. αclA, pclA, spclA, rclA, bclA). A subset A of X is called clopen if it both open and closed in X.

Definition 2.2: A subset A of a space X is called

1) a generalized closed set[12] (briefly, g-closed) if $cl(A)\subset U$ whenever $A\subset U$ and U is open in X.
2) a π-open set[17] if A is finite union of regular open sets.
3) a πg-closed set[6] if $cl(A)\subset U$ whenever $A\subset U$ and U is π-open in X,
4) a πgα-closed set[8] if $\alpha cl(A)\subset U$ whenever $A\subset U$ and U is π-open in X,
5) a πgp-closed set[16] if $pcl(A)\subset U$ whenever $A\subset U$ and U is π-open in X,
6) a πgs-closed set[2] if $scl(A)\subset U$ whenever $A\subset U$ and U is π-open in X,
7) a πgb-closed set[18] if $bcl(A)\subset U$ whenever $A\subset U$ and U is π-open in X,
8) a πgr-closed set[10] if $rcl(A)\subset U$ whenever $A\subset U$ and U is π-open in X.

Definition 2.3 [14]: g-closure of a subset A of X is defined as the intersection of all g-closed sets containing A and g-interior of A is defined as the union of all g-open sets contained in A.

3. π^g-Closed sets

We introduce the following definition:

Definition 3.1: A subset A of X is called π^g-closed set if $gcl(A)\subset U$ whenever $A\subset U$ and U is πg-open in (X, τ).By π^GC(τ), we mean the family of all π^g-closed subsets of the space (X, τ).

Theorem 3.2: Every g-closed set is π^g-closed but not conversely.

Proof: Let A be a g-closed set and $A\subset U$ and U is πg-open. Then $gcl(A) = A\subset U$. Therefore, $gcl(A)\subset U$, whenever $A\subset U$ and U is πg-open. Hence A is π^g-closed.

Converse part is justified in the following example.

Example 3.3: Let X = {a, b, c, d, e} and τ = {ϕ, {a, b}, {c, d}, {a, b, c, d}, X}. Let A = {a}. Then A is π^g-closed but not g-closed.

Proposition 3.4: Every π-closed set is π^g-closed.

Proof: Let A be a π-closed set and A⊂U, U is πg-open. Since πcl(A) = A, gcl(A)⊂ πcl(A) = A, Therefore gcl(A)⊂A whenever A⊂U and U is πg-open. Hence A is π^g-closed.

Proposition 3.5: Every closed set is π^g-closed.

Proof: Let A be a closed set and A⊂U and U is πg-open. Then cl(A) = A⊂U, since A is closed. Since every closed set is g-closed, gcl(A)⊂cl(A)⊂U. Hence A is π^g-closed.

Converse of the above two propositions need not be true is as seen in the following example.

Example 3.6: Let X = {a, b, c, d} and τ = {ϕ, {b}, {c}, {b, c}, {b, d}, {b, c, d}, X}. Let A = {a, b}. Then A is π^g-closed but not π-closed and closed.

Proposition 3.7: Every regular closed set is π^g-closed.

Proof: Let A be regular closed set and A⊂U, U is πg-open in X. Since every regular closed is closed, cl(A)⊂rcl(A).Also every closed set is g-closed, gcl(A)⊂cl(A).Then gcl(A)⊂cl(A)⊂rcl(A) = A, as A is regular closed. Then gcl(A)⊂A⊂U, whenever A⊂U and U is πg-open. Hence A is π^g-closed.

Converse of the above proposition need not be true as seen in the following example.

Example 3.8: In example 3.6, let A= {a}. Then A is π^g-closed but not regular closed.

Theorem 3.9: If A is πg-open and π^g-closed, then A is g-closed.

Proof: Let A be a πg-open and π^g-closed set in X. Let A⊂A, where A is πg-open.Since A is π^g-closed, gcl(A)⊂A whenever A⊂A and A is πg-open. The above implies gcl(A) = A. Hence A is g-closed.

Remark 3.10: The concepts of πgr-closed set and π^g-closed set are independent of each other. It is shown in the following example.

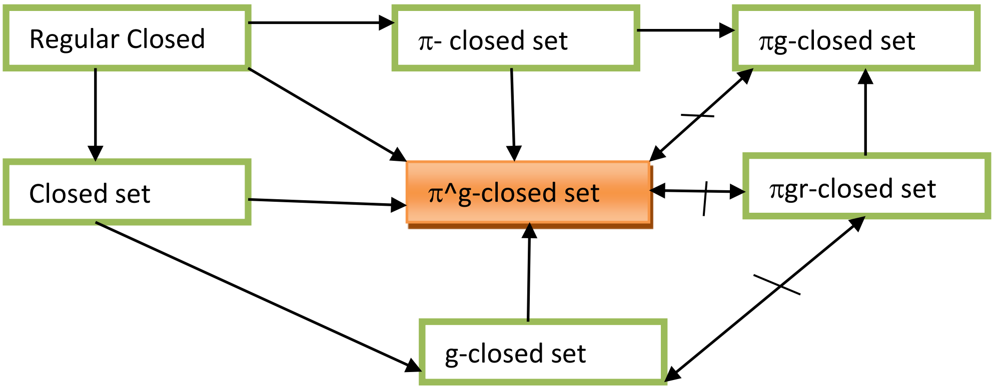
Example 3.11: Let X = {a, b, c, d} and τ = {ϕ, {a}, {b}, {a, b}, {b, d},{a, b, d}, {a, b, c}, X}. Let A = {d}. Then A is π^g-closed but not πgr-closed. Consider B = {b, c}. Then B is πgr-closed but not π^g-closed.

Remark 3.12: The concepts of πg-closed set and π^g-closed set are independent and the same is shown in the following examples.

Example 3.13: Let X = {a, b, c, d} and τ = {ϕ, {a}, {b}, {a, b}, {b, c}, {a, b, c}, {a, b, d}, X}. Let A = {b, d}. Then A is πg-closed but not π^g-closed.

Example 3.14: Let X = {a, b, c, d, e} and τ = {ϕ, {a, b}, {c, d}, {a, b, c, d}, X}. Let A = {a}. Then A is π^g-closed but not πg-closed.

The above discussions are summarized in the following diagram:



Theorem 3.15: Let A be a π^g-closed set in X. Then gcl(A)–A does not contain any non-empty πg-closed set.

Proof: Let F be a πg-closed set such that F⊂gcl(A)–A. Then F⊂X–A implies A⊂X–F. Since A is π^g-closed and X–F is πg-open, gcl(A)⊂X–F. That is, F⊂gcl(A)⋂(X–gcl(A)) =ϕ. This shows that F =ϕ.

Corollary 3.16: Let A be π^g-closed in X. Then A is g-closed if and only if gcl(A)–A is πg-closed.

Proof: ***Necessity:*** Let A be g-closed. Then gcl(A) = A and so gcl(A)–A = ϕ, which is πg-closed.

Sufficiency: Suppose gcl(A)–A is πg-closed. Then gcl(A)–A = ϕ implies gcl(A) = A. This implies A is g-closed.

Theorem 3.17: If A is π^g-closed and A⊂B⊂gcl(A), then B is alsoπ^g-closed.

Proof: Let A be a π^g-closed subset of X such that A⊂B⊂gcl(A). Let U be a πg-open set of X such that B⊂U. Then A⊂U. Since, A is a π^g-closed, we have gcl(A)⊂U. Since B⊂gcl(A), gcl(B)⊂gcl(gcl(A))= gcl(A)⊂U. Then gcl(B)⊂U whenever B⊂U and U is πg-open. Hence B is also π^g-closed set in X.

4. π^g -open sets

Definition 4.1: A set A⊂X is called π^g-open if and only if its complement is π^g-closed.

Remark 4.2: gcl(X–A) = X–gint(A).

Theorem 4.3: Let A⊂X is π^g-open if and only if F⊂gint(A) whenever F is πg-closed and F⊂A.

Proof: ***Necessity:*** Let A be a π^g-open set in X.Let F be πg-closed set and F⊂A. Then X−A⊂X−F, where X−F is πg-open. Since X−A is π^g-closed, gcl(X−A)⊂X−F.By remark 4.2, gcl(X−A)= X−gint(A)⊂X−gint(A)⊂X−F, that is F⊂gint(A).

Sufficiency: Suppose F is πg-closed and F⊂A implies F⊂gint(A).Let X−A⊂U, where U is πg-open. Then X−U⊂A, where X−U is πg-closed. By hypothesis, X−U⊂gint(A).That is, X−gint(A)⊂U. Since gcl(X−A) = X−gint(A), gcl(X−A) ⊂U, where U is πg-open. This implies X−A is π^g-closed and hence A is π^g-open.

Theorem 4.4: If gint(A)⊂B⊂A and A is π^g-open, then B is π^g-open.

Proof: We know that if A is π^g-closed and A⊂B⊂gcl(A),then B is also π^g-closed. Here X−A is π^g-closed, then X−B also π^g-closed. Hence B is π^g-open.

Remark 4.5: For any A⊂X, gint(gcl(A)−A) =ϕ.

Theorem 4.6: If A⊂X is π^g-closed, then gcl(A)−A is π^g-open.

Proof: Let A be a π^g-closed set in X. Let F be a πg-closed set such that F⊂gcl(A)−A.Then gcl(A)−A does not contain any non-empty πg-closed set. Therefore, F=ϕ. So, F⊂gint(gcl(A)−A).This shows gcl(A)−A is π^g-open. Hence A is π^g-closed in (X,τ).

5. Applications

Definition 5.1: A space (X, τ) is called a π^g-$T_{1/2}$ -space if every π^g-closed set is g-closed.

Theorem 5.2: For a topological space (X, τ) the following conditions are equivalent.

1) X is π^g- $T_{1/2}$ -space.
2) Every singleton of X is either πg-closed or g-open.

Proof: (1)⇒(2) : Let x∈X and assume that {x} is not πg-closed. Then clearly X−{x} is not πg-open. Hence X−{x} is trivially a π^g-closed set. Since X is π^g-$T_{1/2}$space, X−{x} is g-closed. Therefore {x} is g-open.

(2)⇒(1) : Assume every singleton of X is either πg-closed or g-open. Let A be a π^g-closed set. Let x∈gcl(A), we will show that x∈A. For that let us consider the following two cases:

Case(1): Let {x} is πg-closed. Then, if x∉A, there exists a πg-closed set in gcl(A)−A. Then {x}∈gcl(A)−A. By theorem3.15, gcl(A)−A does not contain any non-empty πg-closed set and hence {x}∈A. Therefore gcl(A)−A =ϕ and hence gcl(A) = A.

Case(2): Let the set $\{x\}$ be g-open. Since $\{x\}\in gcl(A)$, then $\{x\}\cap gcl(A) = \phi$.The above implies $\{x\}\in A$.So in both cases, $\{x\}\in A$. This shows that $gcl(A)\subset A$ and hence $gcl(A) \neq A$. The above implies $\{x\}\in A$. So in both cases, $\{x\}\in A$. This shows that $gcl(A)\subset A$ and hence $gcl(A) = A$. Therefore A is g-closed.

Theorem 5.3: (i)$GO(\tau)\subset G\pi GO(\tau)$

(ii) A space (X, τ) is π^g- $T_{1/2}$ if and only if $GO(\tau) = \pi$^$GO(\tau)$.

Proof: **(**i) Let A be g-open, then X–A is g-closed. As every g-closed set is π^g-closed, X–A is π^g-closed. Thus A is π^g-open. Hence $GO(\tau)\subset\pi$^$GO(\tau)$

Necessity: Let (X, τ) be π^g-$T_{1/2}$ space. Let $A\in\pi$^$GO(\tau)$.Then X–A is π^g-closed. By hypothesis, X–A is g-closed thus $A\in GO(\tau)$.Thus $GO(\tau) = \pi$^$GO(\tau)$.

Sufficiency: Let $GO(\tau) = \pi$^$GO(\tau)$. Let A be π^g-closed, then X–A is π^g-open implies $X–A\in\pi$^$GO(\tau)$.Hence $X–A\in GO(\tau)$. Hence A is g-closed. This implies (X, τ) is π^g- $T_{1/2}$ space.

6. π^g-Continuous and π^g-irresolute functions

Definition 6.1: A function f: $(X, \tau)\rightarrow(Y, \sigma)$ is called π^g-continuous if $f^{-1}(V)$ is π^g-closed in (X, τ) for every closed set V of (Y, σ).

Example 6.2: Let X = {a, b, c, d} = Y, τ = {ϕ, {a}, {b}, {a, b}, {a, b, c}, X} and σ = {ϕ, {a}, {c}, {a,c}, X}. Define f: $(X,\tau)\rightarrow(Y,\sigma)$ by f(a) = a, f(b) = c, f(c) =d, f(d) = b. Since the inverse image of closed sets in Y are π^g-closed in X, f: $(X, \tau) \rightarrow (Y, \sigma)$ isπ^g-continuous.

Definition 6.3: A function f: $(X,\tau)\rightarrow(Y,\sigma)$ is called π^g-irresolute if $f^{-1}(V)$ is π^g-closed in(X, τ) for every π^g-closed set V in (Y, σ).

Example 6.4: Let X = {a, b, c, d,e}, τ = {ϕ, {a, b}, {c, d}, {a, b, c, d}, X} and Y= {a, b, c, d}, σ = {ϕ, {a}, {b}, {a, b}, {b, d}, {a, b, d}, {a, b, c}, X}. Define f: $(X,\tau)\rightarrow(Y,\sigma)$ by f(a) = b, f(b) = c, f(c) = e, f(d) = a, f(e) =d. Then inverse image of every π^g-closed set is π^g-closed under f. Hence f is π^g-irresolute.

Remark 6.5: Every π^g-irresolute function is π^g-continuous, but not conversely.

Example 6.6: In example 6.4, f is π^g-irresolute and also f is π^g-continuous. In example 6.2, f is π^g-continuous but not π^g-irresolute.

Theorem 6.7: (i) Let f: $(X,\tau)\rightarrow(Y,\sigma)$ be g-continuous. Then f is π^g-continuous.

(ii) Let f: $(X,\tau)\rightarrow(Y,\sigma)$ be regular continuous. Then f is π^g-continuous.

Proof: (i) Let V be a closed set in (Y,σ). Then f^{-1} (V) is g-closed in (X,τ) as f is g-continuous. As every g-closed set is π^g-closed, f^{-1}(V) is π^g-closed. Hence f is π^g-continuous. (ii) Let U be closed in (Y,σ). Then f^{-1} (U) is regular closed in (X, τ) as f is regular continuous. Since every regular closed set is π^g-closed, we have f^{-1} (U) is π^g-closed. Thus f is π^g-continuous. The converse of the above is not true as in the following example.

Example 6.8: Let X = {a, b, c, d, e} = Y, τ = {ϕ, {a, b}, {c, d}, {a, b, c, d}, X} and σ = {ϕ, {a}, {e}, {a, e}, {c, d}, {a, c, d}, {c, d, e}, {a, c, d, e}, X}. Define f: (X, τ)→(Y, σ) by f(a) = e, f(b) = b, f(c) = a, f(d) = c, f(e) = d. Here the inverse image of every closed set in Y is π^g-closed in X but not g-closed in X. Hence π^g-continuous function need not be g-continuous.

The composition of two π^g-continuous functions need not be π^g-continuous and shown in the following example.

Example 6.9: Let X = Y = Z= {a, b, c, d}, τ = {ϕ, {a}, {b}, {a, b}, {a, b, c}, X}, σ = {ϕ, {a}, {c}, {a, c}, X} and η = {ϕ, {a}, {b}, {a, b}, {a, b, d}, X}. Define f: (X, τ)→(Y, σ) by f(a) = a, f(b) = c, f(c) = d, f(d) =b. Define g: (Y, σ)→(Z,η) by g(a) = c, g(b) = d, g(c) = b, g(d) = a. Then f and g are π^g-continuous. But g∘f is not π^g-continuous.

Theorem 6.10: Let f: (X, τ)→(Y, σ) and g: (Y, σ)→(Z, η) be any two functions. Then

i) g∘f is π^g-continuous, if g is continuous and f is π^g-continuous.
ii) g∘f is π^g-irresolute, if g is π^g-irresolute and f is π^g-irresolute.
iii) g∘f is π^g-continuous, if g is π^g-continuous and f is π^g-irresolute.

Proof: (i) Let V be closed in (Z, η).Since g is continuous, g^{-1}(V) is closed in(Y, σ).By π^g-continuity of f implies $f^{-1}(g^{-1}(V))$ is π^g-closed in X.That is, $(g∘f)^{-1}$(V) is π^g-closed in (X, τ).Hence g∘f is π^g-continuous.

(ii) Let V be π^g-closed in (Z, η).Since g is π^g-irresolute, g^{-1}(V) is π^g-closed in (Y, σ).As f is π^g-irresolute, $f^{-1}(g^{-1}(V)) = (g∘f)^{-1}(V)$ is π^g-closed in (X, τ). Therefore g∘f is π^g-irresolute.

(iii) Let V be closed in (Z, η).Since g is π^g-continuous, g^{-1}(V) is π^g-closed in (Y, σ). As f is π^g-irresolute, $f^{-1}(g^{-1}(V))= (g∘f)^{-1}(V)$ is π^g-closed in (X, τ). Therefore g∘f is π^g-continuous.

References:

1) D.Andrijevic, Semi preopen sets, Mat. Vensik 38(1986), 24-32.
2) G.Aslim, A.CascuGuler and T.Noiri, On πgs-closed set in topological spaces, Acta Math. Hungar., 112(4)(2006), 275-283.

3) P.Bhattacharya and B.K.Lahiri, Semi-generalized closed sets in topology, Indian J. Math., 29(1987), 376-382.

4) Dhanya.R and A.Parvathi, On πgb*-closed sets in topological spaces, International Journal of Innovative Research in Science, Engineering and Technology volume 3, may 2014, ISSN: 2319-8753.

5) J.Dontchev and T.Noiri, Quasi-normal spaces and πg-closed sets, Acta Math. Hungar., 89(3)(2000), 211-219.

6) E.Ekici and C.W. Baker, On πg-closed sets and continuity, Kochi J. Math., 2(2007), 35-42.

7) Y. Gnanambal., On Generalized preregular closed sets in topological spaces, Indian J.Pure and appl.math., 28(1997), 351-360.

8) C.Janaki, Studies on πgα-closed sets in topological spaces, Ph.D thesis, Bharathiar University, 2009.

9) Jeyanthi.V and Janaki.C, On πgr-continuous functions, IJERA Volume 3, Jan-Feb 2013, pp.1861-1870.

10) Jeyanthi.V and Janaki.C,Onπgr-closed sets in topological spaces, Asian Journal of current.engg. Maths 1:5 sep 2012, 211-219.

11) N. Levine, Semi-open sets and semi-continuity in topological spaces, Amer.Math.Monthly 70(1963), 36-41.

12) N. Levine, Generalized closed sets in topology, Rend.Circ.Mat.palermo, 19(1970),89-96.

13) H.Maki, R.Devi and K.Balachandran, Associated topologies of generalized α-closed sets and α-generalized closed set, mem.fac.sci.kochiuniv.ser.A math,15(1994),51-63.

14) C. Mugundan and Nagaveni, A weaker form of closed sets, 2011,1949-1961.

15) Palaniappan.N., and Rao.K.C, Regular generalized closed sets, Kyungpook math.J.33(1993), 211-219.

16) Park.J.H , On πgp-closed sets in topological spaces, Acta Mathematica Hungarica, 12, (4),257-283.

17) M.S.Sarsak and V.Rajesh, π-generalised semi-preclosed sets, International mathematical forum, 5(2010), 573-578.

18) D.Sreeja and C.Janaki, On πgb-closed sets in topological spaces, International Journal of Mathematical Archive-2(8),2011, 1314-1320.

19) V.Zaitsev, On certain classes of topological spaces and their bicompactification, Dokl.Aksd.Nauk.SSSR,178(1968), 778-779.